# Application of Constructal Theory to Write Mechanical Maximum Work Principle and Equilibrium State of Continuum Media Flow as a Solution of a Variational Optimization Problem

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***Abstract*—This scientific research proposes a fundamental application of constructal theory developed by prof. Adrian BEJAN of Duke University in order to prove in a mathematical sense that the mechanical maximum work principle used by the theory of continuum media plasticity can be regarded as a solution of a general variational optimization problem. According to the first and second thermodynamics laws, the constructal principle search to complete the natural tendency for all finite-size system to raise the entropy obtaining specific optimal system design or material flow configurations. In accord with the constructal theory "all system searches to flow more and more easily over time using specific distribution of imperfections in order to maximize entropy and to minimize the losses". In this sense, concerning the field of forming processes, all material flows under specified boundaries, loading and processing conditions are those which minimize the sum of dissipated deformation and friction power. Thus all the corresponding mechanical variables (velocities, stress, strain, strain rate) of the real mechanical state as those that minimizes the total dissipated power. It can be then obtained a variational constrained minimization problem. Equivalent form of the maximum work principle is proved also for the friction stresses together with the convexity properties of plastic or friction potential. An application in the case of a cylindrical upsetting shows the feasibility of the proposed minimization problem formulation to find analytical solution. To valid this theory, comparisons are made using the classical analytical analysis based on upper and lower bound theorems, slices method and numerical Finite Element Modelling (FEM).**

***Index Terms*—Constructal theory, continuum media plasticity, maximum work principle, variational optimization.**

## I. INTRODUCTION

Theoretical and applied scientific researches concerning the field of thermodynamic optimization theory for complex systems based on maximization of entropy, minimization of local or global flow resistance and corresponding maximization of system's speed to reach a stable equilibrium state have been developed by Prof. Adrian BEJAN of Duke University. As can be mentioned in previous scientific works [1-2] the named constructal principle of Prof. A. BEJAN can be regarded as a general constrained optimization problem taking into account the principle that "for a finite-size system to persist in time (to live), it must evolve in such a way that it provides easier access to the imposed currents that flow through it". Generally all systems search to optimize imperfections distribution to facilitate the flow and to minimize the local resistance or applied powers. A lot of studies [2] concerning examples from thermal problems, fluid flow or porous media properties, nature phenomena observations, economic or societal behavior confirm this principle. Starting from a recent scientific work of the author this paper proposes to consolidate the general proof of the mechanical maximum work principle [3]. This one is used by the plasticity theory [4] and is generally applied to obtain the governing equations of material flow or strength analysis and to make analytical or numerical computations particularly during forming processes. Previous works [5-6] have search to explain this principle only in a phenomenological way and only for metallic materials starting from local slips of atomic planes. Using the well-known mechanical virtual power principle and based on above mentioned constructal theory the maximum work principle can been obtained from a general variational optimization problem which search to minimize the material flow-resistance or the system dissipations. After detailed theoretical backgrounds together with their useful consequences proving convexity and normal rule of rheological or tribologic potential behaviour, lower-bound and upper-bound theorems, an application is presented concerning a cylindrical quasi-static crushing taking into account a Tresca friction. Starting from theoretical considerations, a validation of proposed techniques to find analytical solutions is also developed using comparisons between the obtained analytical estimations of the loads and the numerical FEM results.

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## II. THEORETICAL BACKGROUNDS

### A. Mathematical Proof of Maximum Work Principle

According to the continuum media theory considering a material flow on a defined body $\Omega$ the Newtonian local mechanical equilibrium balance can be written using the equivalent virtual power works principle (PPW):

$$\int_{\Omega}[\sigma]:\left[\dot{\varepsilon}^*\right]dV+\int_{\partial\Omega}-\vec{\tau}\cdot\Delta\vec{v}^*\,dS=$$
$$\int_{\partial\Omega'}\vec{T}\cdot\vec{v}^d\,dS'+\int_{\partial\Omega''}\vec{T}^d\cdot\vec{v}^*dS''+\int_{\Omega}\rho\vec{f}\cdot\vec{v}^*dV-\int_{\Omega}\rho\frac{d\vec{v}^*}{dt}\cdot v^*\,dV \quad (1)$$

The PPW principle is available for all virtual admissible velocities field $\vec{v}^*$ following boundaries kinematic conditions, where $[\sigma]$ is the Cauchy stress tensor, $\left[\dot{\varepsilon}^*\right]$ is the virtual strain rate tensor defined by $\frac{1}{2}\left\{\left[grad(\vec{v}^*)\right]+\left[grad(\vec{v}^*)\right]^T\right\}$,

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$\vec{\tau}$ is the friction shear vector on contact surface $\partial\Omega$ or shear stress vector of body discontinuities, $\vec{T}$ is the specific load, $\vec{v}^d$ is the imposed velocity on boundary surface $\partial\Omega'$, $\vec{T}^d$ the imposed load on boundary surface $\partial\Omega''$, $\vec{f}$ take into account the mass forces and $\rho$ the material density.

Considering the real velocity field $\vec{v}$ of the material flow the corresponding mechanical power equilibrium becomes:

$$\int_{\Omega}[\sigma]:[\dot{\varepsilon}]dV + \int_{\partial\Omega} -\vec{\tau}\cdot\Delta\vec{v}\,dS = \int_{\partial\Omega'}\vec{T}\cdot\vec{v}^d dS' + \int_{\partial\Omega''}\vec{T}^d\cdot\vec{v}dS'' + \int_{\Omega}\rho\vec{f}\cdot\vec{v}dV - \int_{\Omega}\rho\frac{d\vec{v}}{dt}\cdot\vec{v}dV \tag{2}$$

Using the constructal law in the case of a deformable material or a forming process it can be postulated that under external loading the material flows search to minimize the sum of dissipated powers corresponding to the material deformation, contact friction or surfaces discontinuities and imposed loads. Consequently the real values of all kinematics and mechanical variables (velocities, stresses, strains, strain rate) are those that minimize this dissipation power. So for a set of a virtual variable's state, the real ones must minimize the total dissipation power functional. It can be then concluded that for all plastic materials (metallic or non-metallic) the real strain rate tensor $[\dot{\varepsilon}]$ and the real stress tensor $[\sigma]$ corresponding to the material flow can be obtained by minimizing the sum of dissipated plastic power work, of the friction or surfaces discontinuities power and of the imposed loads corresponding to any virtual state. Furthermore material plasticity is generally governed in terms of stress tensor and friction vector through a plastic loci defined by a scalar multi-variable function $\Phi_p([\sigma])$ =0 together with a similar friction form defined by $\Psi_f(\vec{\tau})$ = 0. In this case, starting from all other admissible virtual velocity field *v** (different for the real one) characterized by a virtual strain rate tensor $[\dot{\varepsilon}^*] \neq [\dot{\varepsilon}]$ and a virtual admissible plastic stress tensor $\lfloor\sigma^*\rfloor$ ($\Phi_p([\sigma^*])$ = 0 and $\Psi_f(\vec{\tau}^*)$ = 0), the real state of the material plastic flow can be obtained from the minimization of a functional defined by the total virtual dissipated power:

$$Min(\dot{W}_d)$$
$$\dot{W}_d = \int_{\Omega}[\sigma^*]:[\dot{\varepsilon}^*]dV + \int_{\partial\Omega} -\vec{\tau}^*\cdot\Delta\vec{v}^*\,dS + \int_{\partial\Omega''} -\vec{T}^d\cdot\vec{v}^*\,dS'' \tag{3}$$

Using boundaries kinematics and loading conditions a constrained variational minimization problem is then obtained and the solution corresponds to the real flow state. In other hand for all above virtual states is available the following inequality:

$$\int_{\Omega}[\sigma]:[\dot{\varepsilon}]dV + \int_{\partial\Omega} -\vec{\tau}\cdot\Delta\vec{v}\,dS + \int_{\partial\Omega''} -\vec{T}^d\cdot\vec{v}\,dS'' \leq \int_{\Omega}[\sigma^*]:[\dot{\varepsilon}^*]dV + \int_{\partial\Omega} -\vec{\tau}^*\cdot\Delta\vec{v}^*\,dS + \int_{\partial\Omega''} -\vec{T}^d\cdot\vec{v}^*\,dS'' \tag{4}$$

For consistent materials and quasi-static conditions, the mass and the inertial forces can be neglected. The PPW principle can be then written in the simplified form:

$$\int_{\Omega}[\sigma]:[\dot{\varepsilon}]dV + \int_{\partial\Omega} -\vec{\tau}\cdot\Delta\vec{v}\,dS + \int_{\partial\Omega''} -\vec{T}^d\cdot\vec{v}\,dS'' = \int_{\Omega}[\sigma]:[\dot{\varepsilon}^*]dV + \int_{\partial\Omega} -\vec{\tau}\cdot\Delta\vec{v}^*\,dS + \int_{\partial\Omega''} -\vec{T}^d\cdot\vec{v}^*\,dS'' \tag{5}$$

Starting from (4) and (5) the following equivalent form is obtained:

$$\int_{\Omega}\left([\sigma^*]-[\sigma]\right):[\dot{\varepsilon}^*]dV + \int_{\partial\Omega'}\left(\vec{\tau}-\vec{\tau}^*\right)\cdot\Delta\vec{v}^*\,dS' \geq 0 \tag{6}$$

This inequality must be available for any virtual stress or friction state ($\Phi_p([\sigma^*])$ = 0 and $\Psi_f(\vec{\tau}^*)$ = 0) and specified finite size material domains. Consequently each term of integrals must have positive values i.e.:

$$\left([\sigma^*]-[\sigma]\right):[\dot{\varepsilon}^*] \geq 0, \forall[\sigma^*], \Phi_p([\sigma^*]) = 0 \tag{7}$$

and

$$-\left(\vec{\tau}^*-\vec{\tau}\right)\cdot\Delta\vec{v}^* \geq 0, \forall\vec{\tau}^*, \Psi_f(\vec{\tau}^*) = 0 \tag{8}$$

In a reverse form, taking into account the real plastic flow characterized by the velocity field *v*, the strain rate tensor $[\dot{\varepsilon}]$ and the stress state $[\sigma]$, all other admissible stress state $\lfloor\sigma^*\rfloor$ generally written at the sum of an equilibrium stress and a pressure without plastic work dissipation, must verify:

$$\left([\sigma]-[\sigma^*]\right):[\dot{\varepsilon}] \geq 0, \forall[\sigma^*], \Phi_p([\sigma^*]) = 0 \tag{9}$$

And

$$-\left(\vec{\tau}-\vec{\tau}^*\right)\cdot\Delta\vec{v} \geq 0, \forall\vec{\tau}^*, \Psi_f(\vec{\tau}^*) = 0 \tag{10}$$

Using the above relationships (7), (8), (9) and (10) it can be proved the convex form of the functions defining plastic and friction criteria and the known normal rule properties. So the plastic strain rate must be proportional to the stress component's gradient:

$$[\dot{\varepsilon}] = \lambda_p \partial\Phi_p / \partial[\sigma], \lambda_p \geq 0 \tag{11}$$

In the same way the friction law can be written using the following general form:

$$\Delta\vec{v} = -\lambda_f \partial\Psi_f / \partial\vec{\tau}, \lambda_f \geq 0 \tag{12}$$

In this case the two conditions expressed by (9) and (10) can be extended to virtual stresses defined by $\Phi_p([\sigma^*]) \leq 0$ and $\Psi_f(\vec{\tau}^*) \leq 0$. It is then possible to conclude that for all virtual stress state:

$$\left([\sigma]-[\sigma^*]\right):[\dot{\varepsilon}] \geq 0, \forall[\sigma^*], \Phi_p([\sigma^*]) \leq 0 \tag{13}$$

And

$$-\left(\vec{\tau}-\vec{\tau}^*\right)\cdot\Delta\vec{v} \geq 0, \forall\vec{\tau}^*, \Psi_f(\vec{\tau}^*) \leq 0 \tag{14}$$

The first inequality it is known in the plasticity theory as the maximum work principle [4-6] and is proved here that this one can be obtained as a consequence of the virtual power works and of the constructal theory principles [1], [3]. Moreover using the above theoretical proof it can be concluded that the maximum work principle can be applied for all type of continuum media: fluids, solid, mushy state, metallic or non-metallic materials and also for the friction stresses state. Specified analytical expressions of convex plastic or friction criteria $\Phi_p([\sigma]) = 0, \Psi_f(\vec{\tau}) = 0$ can be used to define isotropic or anisotropic behavior of plastic materials, respectively of interfaces slide laws.

*B. Practical Considerations*

Starting from the maximum work principle it can be obtained the lower and upper bounds theorems (Fig. 1) [4-6]. These ones can be also regarded that variational optimization formulations and are frequently employed to obtain analytical estimations of loads during material forming processes under particular hypothesis. Are also used to compare and valid the results obtained by Numerical Methods as the FEM.

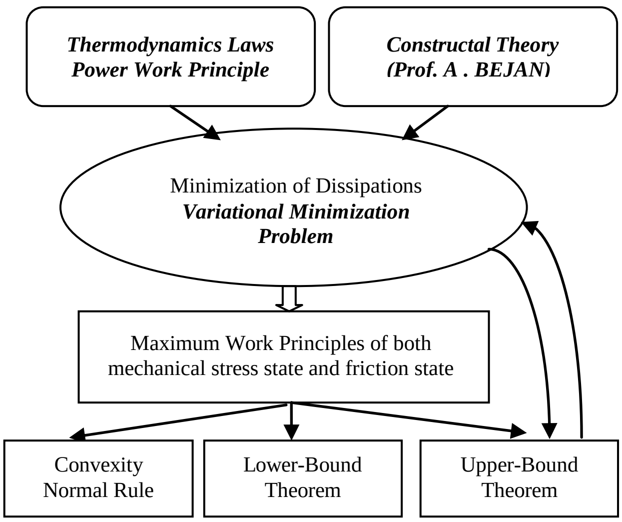


Fig. 1. Synthesis of proposed proof of Maximum Work Principle starting form Virtual Power Work Principle (PPW) and the variational minimization problem formulation based on constructal theory [3].

Regarding the Upper-Bound Theorem [6] an equivalent form of minimization variational problem (3) is obtained. It can be then concluded on the coherency and on the equivalence between the maximum work principle and the principle of minimization of dissipations obtained from the constructal theory.

## III. Application to Cylindrical Upsetting Test

As compared to the previous work of the author [3] in this paper it is proposed to apply the previous variational optimization problem (3) in the particular case of a cylindrical quasi-static upsetting process [7-8]. So a cylindrical sample of initial height $2H_0$ and initial diameter $2R_0$ (with $H_0 \le (1.5 \div 3)R_0$ to avoid buckling phenomenon) is crushed between two flat heaps by a simple operation carried out on a double-action hydraulic press with constant vertical speed $V$ (Fig. 2).

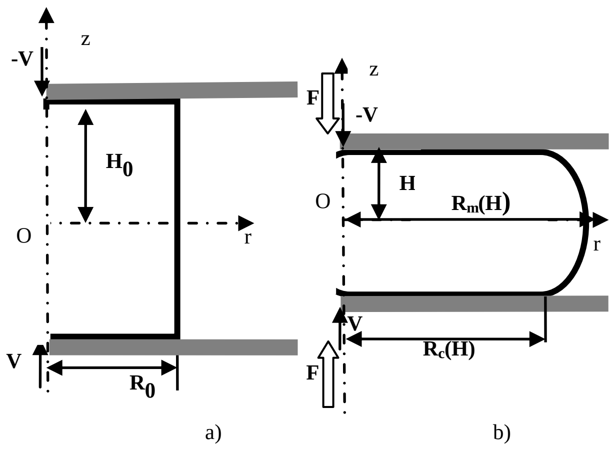


Fig. 2. Quasi-static cylindrical upsetting test: a) Axisymmetric sample geometry and kinetic configuration, b) Crushing configuration at the time *t*.

At any time *t*, taking into accounts the friction phenomenon, the dimensions will be $2H$ over the height, $2R_c$ over the width of the area of contact with the flat tools and $2R_m$ for the maximum diameter. It is supposed that the material is incompressible and has a perfect rigid plastic behavior defined by a Von-Mises criterion used to define the equivalent plastic stress $\bar{\sigma} = \sigma_0$. Concerning the friction at the workpiece-tool interfaces a Tresca model can be used defining the friction shear by $\tau = \bar{m}\sigma_0 / \sqrt{3}$. Taking into account the axisymmetric compression, the material incompressibility and all the boundaries conditions, a virtual admissible incompressible velocity field $v^*$ with $div(v^*) = \dot{\varepsilon}_{rr} + \dot{\varepsilon}_{\theta\theta} + \dot{\varepsilon}_{zz} = 0$ can be defined only in a quarter part of the material specimen ($0 \le z \le H$ and $0 \le r \le R_m(H) \approx R_c(H) \approx R(H) \approx R_0\sqrt{H_0 / H}$) using the following expressions [7]:

$$\begin{aligned} & v_r^*(r,z) = AV\frac{r}{H}\exp(-\beta z/2H) \\ & v_z^*(z) = 4AV\left[\exp(-\beta z/2H) - 1\right]/\beta \\ & v_\theta^*(r,z) = 0 \\ & v_z^*(0) = 0, v_z^*(H) = -V \end{aligned} \quad (15)$$

Here $A = \beta[1 - exp(-\beta/2)]/4$ where β is a shape parameter. Taking into account that $\dot{\varepsilon}_{rr} = \dot{\varepsilon}_{\theta\theta}$ and $\dot{\varepsilon}_{zz} = -2\dot{\varepsilon}_{rr}$ the corresponding virtual equivalent strain rate $\dot{\bar{\varepsilon}}^*(r,z) = \sqrt{\frac{2}{3}\left(\dot{\varepsilon}_{rr}^2 + \dot{\varepsilon}_{\theta\theta}^2 + \dot{\varepsilon}_{zz}^2 + 2\dot{\varepsilon}_{rz}^2\right)} = \frac{2}{\sqrt{3}}\sqrt{\left(3\dot{\varepsilon}_{rr}^2 + \dot{\varepsilon}_{rz}^2\right)}$ is then defined by:

$$\dot{\bar{\varepsilon}}^*(r,z) = \frac{1}{2\sqrt{3}}\frac{V}{H}\frac{\beta}{\left[1 - exp(-\beta/2)\right]} exp(-\beta z/2H)\sqrt{3 + (1/4)(\beta/2H)^2 r^2} \quad (16)$$

Concerning the sum $\dot{W}_d^*$ of dissipated plastic power work and friction power this one can be estimated from:

$$\dot{W}_d^* = 2\int_0^H \int_0^R \sigma_0 \dot{\bar{\varepsilon}}^* 2\pi r dr dz + 2\int_0^R \left|\tau \Delta v^*\right| 2\pi r dr \quad (17)$$

where $0 \le r \le R(h) \approx Rc, 0 \le z \le H$ , $\left|\Delta v^*\right| = v_r^*(r,H)$

Using the notation $\alpha(\beta) = 2H / \beta R$ it can be shown that:

$$\dot{W}_d^* = 2\pi\sigma_0 VR^2 \left[ \phi[\alpha(\beta)] + \bar{m}\varphi(\beta)(R/2H)/3\sqrt{3} \right] \quad (18)$$

The shape functions $\phi[\alpha(\beta)]$ and $\varphi(\beta)$ have the forms:

$$\phi\left[\alpha(\beta)\right] = 8\alpha(\beta)^{-1}\left[\left(\frac{1}{12}+\alpha(\beta)^2\right)^{3/2} - \left(\alpha(\beta)^2\right)^{3/2}\right] \quad (19)$$
$$\varphi(\beta) = \beta\exp(-\beta/2)/\left[1-\exp(-\beta/2\right]$$

Using the variational minimization principle formulated by equation (3) an estimation of the real velocity field can be obtained by minimization of parametric functions:

$$\dot{W}_d \le Min\, \dot{W}_d^*(\beta) \quad (20)$$

It is then required to find $\beta_{opt}$ that minimize $\left\lfloor \phi[\alpha(\beta)] + \bar{m}\varphi(\beta)(R/2H)/3\sqrt{3} \right\rfloor$ i.e. solution of the non-linear equation:

$$\frac{d\phi}{d\beta} + \bar{m}\frac{d\varphi}{d\beta}(R/2H)/3\sqrt{3} = 0 \quad (21)$$

An analytical approximation of the real velocity field can be then obtained from the optimal value $\beta_{opt}$ solution of equation (18). As has been shown in [8] an approximate form can be written by $\beta_{opt} \approx (4\bar{m}/\sqrt{3})/\left\lfloor (R/2H) + 2\bar{m}/3\sqrt{3} \right\rfloor$. Concerning the real upsetting load $F$ this one must verifies:

$$F \le \tilde{F} = \frac{\dot{W}_d^*(\beta_{opt})}{2V} \quad (22)$$

Because the calculation of the force for this optimal solution is quite complex it can be observed from the minimization problem (20) that for all values of parameter $\beta$ the real dissipated work $\dot{W}_d \le \dot{W}_d^*(\beta)$. In the particular case where $\beta \to 0$ i.e. $\alpha(\beta) \to \infty$ it can be shown that $\phi\left\lfloor \alpha(\beta) \right\rfloor \to 1$ and $\varphi(\beta) \to 2$. The corresponding value of the virtual dissipated power becomes:

$$\dot{W}_d^*(0) = 2\pi R^2 V\left[1 + 2\bar{m}(R/2H)/3\sqrt{3}\right] \quad (23)$$

Consequently the real upsetting load verifies the following condition:

$$F \le \tilde{F} = \frac{\dot{W}_d^*(0)}{2V} = \pi\sigma_0 R^2\left[1 + 2\bar{m}(R/2H)/3\sqrt{3}\right] \quad (24)$$

Despite the simplified form of this obtained load estimation $\tilde{F}$ it can be observed also that this one has the same expression such as that obtained from the slices method computation [5] based on a mean mechanical equilibrium written along the $r$ axis by $d\sigma_{rr}/dr = \tau/H$, on a global mean value of $\tau_{rz}$ equal to 0, of the plastic criterion defined by $|\sigma_{rr} - \sigma_{zz}| = \sigma_0$, $\sigma_{zz}(r) = -\sigma_0 + (r-R)\bar{m}\sigma_0/2H\sqrt{3}$ and of the axial forces evaluation by $\tilde{F} = -\int_0^R \sigma_{zz}(r)2\pi r dr$ .

It can be also noted that this result can be regarded that as optimal upper bound estimation obtained from PPW equation and results of equation (13) regarded as a consequence of the constructal theory expressed quantitatively by the solution of the general minimization dissipated power functional (3). In the same way using the proved maximum work principle written by equation (9), obtained as a consequence of a reverse form of the optimal problem (3), it can be obtained a lower bound estimation of the real load. In this case it is chosen a virtual stress state [$\sigma$*] verifying the mechanical equilibrium equations with $\bar{\sigma}^* \le \sigma_0$ ( $\Phi_p\left(\left[\sigma^*\right]\right) \le 0$ ) and the Tresca friction law ( $\tau = \bar{m}\sigma_0/\sqrt{3}$ ) on the specimen-tool contact ( $\Psi_f\left(\vec{\tau}^*\right) = \Psi_f\left(\vec{\tau}\right) = 0$ ). So the following expressions are proposed to define a admissible static virtual stress components ( $Div\left\lfloor \sigma^* \right\rfloor = 0$ ) corresponding to the symmetric specimen part ( $r \ge 0, z \ge 0$ ):

$$\begin{aligned} \sigma_{rr}^* &= \sigma_{\theta\theta}^* = (r-R)\tau/H \\ \sigma_{zz}^* &= -\sigma'_0 + (r-R)\tau/H \\ \sigma_{rz}^* &= -\tau z/H, \sigma_{r\theta}^* = 0, \sigma_{\theta z}^* = 0 \end{aligned} \quad (25)$$

Because this virtual stress state verifies the mechanical equilibrium it is possible to write the PPW principle i.e.:

$$\begin{aligned} &\int_\Omega\left[\sigma^*\right]:\left[\dot{\varepsilon}^*\right]dV + \int_{\partial\Omega'}\bar{m}\sigma_0\cdot\Delta\vec{v}^*/\sqrt{3}dS' + \int_{\partial\Omega'}-\vec{T}^*\cdot\vec{v}^d\,dS' + \\ &\int_{\partial\Omega''}-\vec{T}^{*d}\cdot\vec{v}^*\,dS'' = \int_\Omega\left[\sigma^*\right]:\left[\dot{\varepsilon}\right]dV + \int_{\partial\Omega}\bar{m}\sigma_0\cdot\Delta\vec{v}/\sqrt{3}dS + \\ &\int_{\partial\Omega'}-\vec{T}^*\cdot\vec{v}^d\,dS' + \int_{\partial\Omega''}-\vec{T}^{*d}\cdot\vec{v}\,dS'' = 0 \end{aligned} \quad (26)$$

In the same way the real stress state [$\sigma$] verifies the mechanical power equilibrium written in the form ( $\vec{T}^d = 0$ ):

$$\int_\Omega[\sigma]:[\dot{\varepsilon}]dV + \int_{\partial\Omega}\bar{m}\sigma_0\cdot\Delta\vec{v}/\sqrt{3}dS + \int_{\partial\Omega'}-\vec{T}\cdot\vec{v}^d\,dS' = 0 \quad (27)$$

Starting from the relationships (9), using the kinematics boundaries conditions (Figure 2) and taking into account from (25) that $\vec{T}^{*d}\vec{v} \ge 0$ it can be observed that equations (26) and (27) give $2V\int_0^R \vec{T}^*(-\vec{r})2\pi r dr \le 2FV$ .

Using the Von-Mises criterion together with the condition of $\Phi_p\left(\left[\sigma^*\right]\right) = \bar{\sigma} - \sigma_0 \le 0$ it is necessary to have:

$$\sqrt{\sigma_0'^2 + \bar{m}^2\sigma_0^2 z^2/H^2} \le \sigma_0, \forall z \in [-H, H] \quad (28)$$

By choosing $\sigma'_0 = \sigma_0\sqrt{1-\bar{m}^2}$ it is obtained:

$$2V\int_0^R \vec{T}^*(-\vec{z})2\pi r dr = 2V\int_0^R -\sigma_{zz}^*(H)2\pi r dr = 2\pi R^2 V\sigma_0\left(\sqrt{1-\bar{m}^2} + 2\bar{m}(R/2H)/3\sqrt{3}\right) \quad (29)$$

Finally the real load $F$ verifies:

$$F \geq \tilde{\tilde{F}} = \pi R^2\sigma_0\left(\sqrt{1-\bar{m}^2} + 2\bar{m}(R/2H)/3\sqrt{3}\right) \quad (30)$$

The optimal upper and lower bounds of load values expressed by the relationships (24) and (30) give an error estimation of the real load around of $(1-\sqrt{1-\bar{m}^2})/\left(1+2\bar{m}(R/2H)/3\sqrt{3}\right)$ where $\bar{m}\in[0,1]$ and permits an average estimation of the load by $\bar{F} \approx (\tilde{F} + \tilde{\tilde{F}})/2$ where:

$$\bar{F} = \pi R^2\sigma_0\left[\frac{1}{2}\left(1+\sqrt{1-\bar{m}^2}\right) + 2\bar{m}(R/2H)/3\sqrt{3}\right] \quad (31)$$

Concerning the case of a compression without friction ($\bar{m}=0$) we obtain $\tilde{F} = \tilde{\tilde{F}}$ and consequently the real load is defined by $F = \pi R^2\sigma_0$. This expression is identical to the analytical result obtained from local incompressibility condition (defining the velocity field by $v_r(r) = Vr/H$, $v_z(z) = -Vz/H$), static momentum balance, plastic normal rule and rigid plastic material behavior (obtaining a diagonal stress tensor defined by $\sigma_{rr}=0, \sigma_{\theta\theta}=0, \sigma_{zz}=-\sigma_0$) [6]. Furthermore in this case $\beta = 0$ gives practically the global minimum of expression (18) and proves then that the material flow solution is obtained from the optimization problem (3).

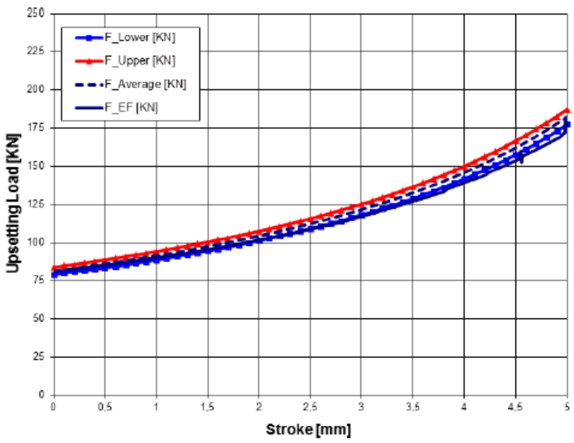


Fig. 3. FEM validation concerning lower/upper estimations of the stroke-load variation in the case of a rigid-plastic material's cylindrical upsetting.

Fig. 3 syntheses the comparisons of previous lower, upper and average estimations of the load variation with results obtained from a Finite Element Modelling (FEM) using the commercial software Forge2® for a cylindrical upsetting of a specimen with $R_0$ = 10 mm, $H_0$ = 10 mm, $\sigma_0$ = 250 MPa, $V$ = 1 mm/s and $\bar{m}$ = 0.35.

## IV. Conclusion

Along the content of this paper has been proved that for a plastic flow of any continuum media, starting from a general principle concerning the minimization of dissipated energy characterizing the evolution of a finite size material body as described by constructal theory, can be finding in a more general form the maximum work principle both for the bulk stress and contact friction. Application on a cylindrical compression of a rigid plastic material shows the feasibility of the proposed minimization problem formulation to find analytical estimations. Comparisons between optimal upper, lower, average loads estimations and Finite Element Simulation have shown the high accuracy of obtained analytical estimations.

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