Constrained-Optimal Based Loop-Shaping State Feedback Approach for Missile Autopilot Design

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***Abstract*—Modern tactical missile systems are required to achieve high maneuverability and sufficient stability. As a result, the flight control design of a missile system is a creative process as it considers both performance requirements and performance limitations, which are inherently conflicting. Different modern control techniques handle such conflicting demands through the adjustment of cost weighting parameters between system internal states and control signals. The adjustment processes might need trials to tune the system performance to certain level at different operating conditions. This paper involves in the formulation of an optimal design approach that achieves the required level of robustness related to open-loop design requirements and system dynamic limits while minimizing the tracking error between the reference input and the system output. The proposed approach is based on a constrained optimization technique where the design parameters are automatically adjusted to the optimum tradeoff between the overall system performance and robustness. The effectiveness and feasibility of the proposed approach are demonstrated through a numerical example for the three-loop autopilot design.**

***Index Terms*—Optimal-robust state feedback, constraint optimization, frequency domain constraint, control effort constraint.**

I. Introduction

The modern techniques such as H∞, μ-synthesis [1], optimal LQG [2] and dynamic inversion [3] are the most popular in the design of autopilot systems for several decades. While these techniques offering powerful design tools, they also suffer from certain shortcomings when put to practice. For example, these methods solve the tradeoff between design performance and robustness requirements indirectly by using weighting parameters on the internal states and the control signal of the system. Since the relation between these weightings and the resultant performance is not so clear, the selection and adjustment of these weightings and some other design parameters might be repeated at different operating conditions to meet the design requirements. Moreover, the autopilots obtained with these techniques are mostly of high order, which is may be difficult to implement.

In general, the system achieves better in terms of time performance criteria as more as the tracking error minimized. However, the free minimization of tracking error may cause too high autopilot gain with an undesired frequency response. Consequently, the optimal control method should provide the optimum of the tracking performance combined with a direct incorporation of the frequency-domain criteria values and actuator limits to achieve a satisfactory robustness level. In the same sense, the minimum error between the desired and the actual open-loop crossover frequency is formed as an objective of LQR technique while adjusting its weightings [4], [5]. However, this method is based on initial guessing of the weightings which might need to be carried out and repeated to adjust the required initial performance. Besides, this scheme will not essentially guarantee an optimal autopilot as it is possible to get the same crossover frequency for different gain designs. Moreover, [6] introduces the multi-objectives optimization technique where both time and frequency performance are combined into one objective function through multi-weight technique. However, it is yet facing the burden of objective's weightings adjustment. The autopilot design using constraint optimization methods is introduced in [7], [8]. These methods are considered for specified controller structure with totally numerical procedure.

The main objective of this work is the formulation of an optimal control technique that achieves the optimum of the performance objective within direct constraints upon the robustness requirements and system limits in such a way that the design solution is automatically obtained without the requirement for weighting parameters [9]. In addition, this formulation allows a direct incorporation of the design requirements that cannot be handled easily using the quadratic performance index such as shaping of loop frequency responses. These requirements are provided through the constraints or the bounds on the design parameters. This paper is organized as follows; Section II discusses the system modeling and transforms the design problem into the parameters of the desired closed-loop diagonal form. Besides, this section derives a formula between these parameters and the corresponding feedback gain. In Section III, the optimal autopilot design objective is formed analytically for the tracking performance based on the integral square error (ISE) index [10], [11]. Moreover, the performance constraints on the open-loop performance and the actuator limits are derived in the form of inequality functions. Furthermore, the method of solving the design problem within the frame of the constrained optimization technique is highlighted. Finally, Section IV shows a numerical autopilot design of a typical missile system using the proposed technique.

II. Modeling and Transformation

Consider a linear time invariant system described by

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$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re$ is the controlled input signal, $y(t) \in \Re$ is the system output and the A, B and C are $n \times n$, $n \times 1$ and $n \times 1$ constant matrices respectively. The feedback control law for controllable (A, B) with available full state, shown in Fig. 1, is in the form of

$$u(t) = -Kx(t) + K'_{DC} v(t) \tag{2}$$

where $v(t) \in \Re$ is the reference input, K is a $1 \times n$ constant feedback gain and K'_{DC} is an pre-adjustment gain for zero steady state error between the input command $v(t)$ and the output $y(t)$.

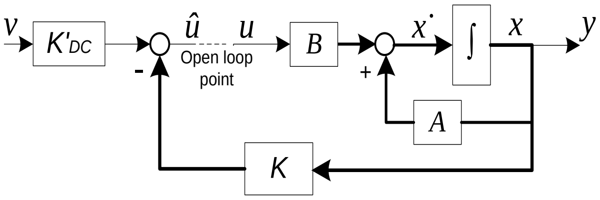


Fig. 1. State feedback system.

The closed-loop dynamics of the system is given as

$$\begin{aligned}\dot{x}(t) &= (A - BK)x(t) + Bv'(t) \\ y(t) &= Cx(t)\end{aligned} \tag{3}$$

The gain K should be chosen properly such that the matrix $A_{cl} = (A - BK)$ is asymptotically stable with a satisfactory transient response. Stable closed-loop systems for different K could be described by a given (Λ, E) in the eigenvalue-eigenvector spectrum, i.e.,

$$(A - BK)E = E\Lambda \tag{4}$$

where Λ is a diagonal matrix of distinct left hand side eigenvalues and E is the corresponding full linear independent eigenvector matrix. The system closed-loop stability and performance are specified by these eigenvalues [12]. However, direct adjustment of the gain K to satisfy system stability and performance may be not an easy task. Instead, the approach in this paper will start from a desired matrix Λ. Then the gain K is calculated based on Eq. (4). The desired Λ consists of n poles in the terms of positive parameters τ, ζ_i and ω_j such that

$$\lambda_1 = -\tau^{-1},\ \lambda_i = -\zeta_{i/2}\omega_{i/2} + j\omega_{i/2}\sqrt{1-\zeta_{i/2}^2},\ \lambda_{i+1} = \lambda_i^*$$

In case of n is odd integer and $i = 2, 4, \ldots (n-1)$, or

$$\lambda_i = -\zeta_{(i+1)/2}\omega_{(i+1)/2} + j\omega_{(i+1)/2}\sqrt{1-\zeta_{(i+1)/2}^2},\ \lambda_{i+1} = \lambda_i^*$$

for even integer n and $i = 1, 3, \ldots (n-1)$. The total number of the positive parameters τ, ζ_i and ω_j is always equal to n. The configuration of matrix Λ guarantees a stable closed-loop. Moreover, the values of these parameters reflect the performance behavior of the system. At this point, it is required to solve matrix E; this task could be accomplished either in term of the parameters of matrix A_{cl} or matrix Λ. In the following, the design will be handled in the space of the positive parameters τ, ζ_i and ω_j. Moreover, a method to find the matrix E in the term of these design parameters is introduced.

Let the diagonal form of a linear dynamic system with distinct roots is given by [13]

$$\begin{aligned}\dot{\bar{x}}(t) &= \Lambda\bar{x}(t) + \bar{B}K'_{DC}v(t) \\ y(t) &= \bar{C}\bar{x}(t)\end{aligned} \tag{5}$$

This form is an equivalent state-space description of the closed-loop dynamics in Eq. (3) under a linear state transformation such that

$$x(t) = E\bar{x}(t),\ \ \bar{B} = E^{-1}B = [1, 1, \ldots 1]^T,\ \ \bar{C} = CE.$$

In order to find the matrix E, a further insight into the system behavior is provided by the transfer function between $\bar{x}(t)$ and $v(t)$ which is given as

$$\bar{T}(s) = K'_{DC}(sI - \Lambda)^{-1}\bar{B} \tag{6}$$

where $\bar{T}(s)$ is an $n \times 1$ transfer-matrix. Using the linear transformation E, the transfer function between $x(t)$ and $v(t)$ is written as

$$\begin{aligned}T(s) &= K'_{DC}E(sI - \Lambda)^{-1}\bar{B} \\ &= K'_{DC}\frac{1}{|sI - \Lambda|}E\bar{q}(s)\bar{B}\end{aligned} \tag{7}$$

where $T(s)$ is an $n \times 1$ transfer-matrix, $|sI - \Lambda|$ is the closed-loop characteristic polynomial of n degree, $\bar{q}(s) = \operatorname{adj}(sI - \Lambda)$ is an $n \times n$ polynomial matrix, its entries are polynomials of $n-1$ degree or smaller, both $\bar{q}(s)$ and $|sI - \Lambda|$ are expressed respectively as

$$\begin{aligned}\bar{q}(s) &= \bar{q}_{n-1}s^{n-1} + \bar{q}_{n-2}s^{n-2} + \cdots + \bar{q}_1 s + \bar{q}_0 \\ |sI - \Lambda| &= s^n + p_{n-1}s^{n-1} + \cdots + p_1 s + p_0\end{aligned} \tag{8}$$

where $p_0, \ldots, p_{n-1}$ are constant coefficients and $\bar{q}_0, \ldots, \bar{q}_{n-1}$ are $n \times n$ constant matrices; both are defined by τ, ζ_i and ω_j. Leverrier-Fadeev method is used to evaluate these matrices as [13]

$$\begin{aligned}\bar{q}_{n-1} &= I_n \\ p_i &= -(1/(n-i))\operatorname{tr}(\Lambda\bar{q}_i) \\ \bar{q}_{i-1} &= \Lambda\bar{q}_i + p_i I_n\end{aligned} \tag{9}$$

where $i = n-1, n-2, \ldots, 1$. By performing Eq. (9), it is found that $\bar{q}_0, \ldots, \bar{q}_{n-1}$ are diagonal matrices and are equal to

$$\bar{q}_{n-1} = I_n,$$

$$\bar{q}_{n-i} = \begin{bmatrix} \bar{q}_{(n-i)_1} & 0 & \cdots & 0 \\ 0 & \bar{q}_{(n-i)_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{q}_{(n-i)_n} \end{bmatrix} \tag{10}$$

where

$$\begin{aligned} \bar{q}_{(n-i)_j} &= \lambda_j^{i-1} + \lambda_j^{i-2} p_{n-1} + \lambda_j^{i-3} p_{n-2} + \ldots + p_{n-(i-1)} \\ &= \lambda_j^{i-1} + \sum_{l=0}^{i-2} \lambda_j^l p_{n-(i-1-l)} \end{aligned},$$

$i = 2,3,\cdots,n$ and $j = 1,2,\cdots,n$. The transfer function matrix $T(s)$ could be reformed as

$$\begin{aligned} T(s) &= K'_{DC} \frac{1}{\prod_{i=1}^{n}(s-\lambda_i)} E\left[\bar{Q}_s(s)\right]_{n\times 1} \\ &= K'_{DC} \frac{1}{\prod_{i=1}^{n}(s-\lambda_i)} E\bar{Q}S \end{aligned} \tag{11}$$

where $\bar{Q}_s(s) = \bar{q}(s)\bar{B}$ is an $n\times 1$ polynomial matrix, $\bar{Q}$ is an $n\times n$ constant matrix and $S = [s^{n-1}\ s^{n-2}\ \cdots 1]^T$. The elements of $\bar{Q}$ are function of τ , ζ_i and ω_j such that $\bar{Q}$ equals

$$\bar{Q} = \begin{bmatrix} 1 & \bar{q}_{(n-2)_1} & \bar{q}_{(n-3)_1} & \cdots & \bar{q}_{(0)_1} \\ 1 & \bar{q}_{(n-2)_2} & \bar{q}_{(n-3)_2} & \cdots & \bar{q}_{(0)_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \bar{q}_{(n-2)_n} & \bar{q}_{(n-3)_n} & \cdots & \bar{q}_{(0)_n} \end{bmatrix} \tag{12}$$

It can be seen that the $\bar{Q}$ is a nonsingular and invertible matrix for positive selections of τ , ζ_i and ω_j , which lead to linearly independent columns and rows.

In the same way, the transfer matrix $T(s)$ could be reproduced using the system description in Eq. (3) as

$$\begin{aligned} T(s) &= K'_{DC}\left(sI - A_{cl}\right)^{-1} B \\ &= K'_{DC} \frac{1}{|sI - A_{cl}|} q(s)B \end{aligned} \tag{13}$$

where $q(s) = \text{adj}(sI - A_{cl})$ is an $n\times n$ polynomial matrix, which is evaluated in the same way as $\bar{q}(s)$, and it is expressed as $q(s) = q_{n-1}s^{n-1} + \cdots + q_1 s + q_0$.

Theorem 1 The linear time invariant system that is described by Eq. (1) and is controlled by the state feedback control in Eq. (2), has the following property

$$adj(sI - A + BK)B = adj(sI - A)B .$$

Proof is given in the Appendix.

Consequently, the matrixes $q_0,\ldots,q_{n-1}$ are calculated by Leverrier-Fadeev algorithm using the system original matrix A . Eq. (13) is reintroduced as

$$\begin{aligned} T(s) &= K'_{DC} \frac{1}{|sI - A_{cl}|}\left[Q_s(s)\right]_{n\times 1} \\ &= K'_{DC} \frac{1}{|sI - A_{cl}|} QS \end{aligned} \tag{14}$$

where $Q_s(s) = q(s)B$ is an $n\times 1$ polynomial matrix that is treated similarly as in Eq. (11) such that Q is an $n\times n$ constant matrix, and its elements are function in the original system parameters and are given by

$$Q_{ij} = \sum_{l=1}^{n} B_l q_{(n-j)il} .$$

In fact, the two equations (11) and (14) are identical as the transfer $T(s)$ is invariant under a linear transformation. Similarly, the characteristic polynomial is also invariant, which results with $E\bar{Q} = Q$. In Common, for any chosen value of the positive design parameters τ , ζ_i and ω_j the corresponding E matrix is found as

$$E(\tau,\zeta_i,\omega_j) = Q\bar{Q}^{-1}(\tau,\zeta_i,\omega_j) \tag{15}$$

where $\bar{Q}$ is a nonsingular matrix. After evaluating the matrix E , the autopilot gain could be expressed in terms of the design parameters as

$$K(\tau,\zeta_i,\omega_i) = B^{\dagger}\hat{A}(\tau,\zeta_i,\omega_i) \tag{16}$$

where $\hat{A} = (A - E\Lambda E^{-1})$, $(B^T B)^{-1}$ is nonsingular square matrix and $B^? = B^T(B^T B)^-$ is the pseudo-inverse of matrix B .

As noted above, the K'_{DC} gain is provided to achieve zero steady state error between the input $v(t)$ and the output $y(t)$. To this end, the corresponding transfer function is given as

$$\begin{aligned} \hat{T}(s) &= \frac{y(t)}{v(t)} = K'_{DC}C\left(sI - A_{cl}\right)^{-1} B \\ &= K'_{DC} \frac{1}{|sI - A_{cl}|} Cq(s)B \end{aligned} \tag{17}$$

Referring to Eq. (11) and (14), the transfer function $\hat{T}(s)$ is reformed as

$$\hat{T}(s) = K'_{DC} \frac{1}{\prod_{i=1}^{n}(s-\lambda_i)} CQS \tag{18}$$

The steady state error tends to zero when

$$\lim_{s\to 0}\hat{T}(s) = 1,\ i.e.,\ K'_{DC} = \frac{p_0}{Q_0} = \frac{\prod_{i=1}^{n}\lambda_i}{\sum_{i=1}^{n} C_i Q_{in}} \tag{19}$$

where p_0 is the free parameter in $|sI - \Lambda|$ calculated by Vieta's formulas [14], C_i are elements of matrix C and Q_0

is assumed to be nonzero for practical system parameters.

III. Autopilot Design

One important problem is concerned with the performance level that can be achieved by the controlled system. Obviously, the feedback design requirements are inherently competitive, so a tradeoff must be performed among different design requirements [15]. Such tradeoff could be fairly handled by discriminating these requirements into performance objective requirements and performance limits. Meanwhile, a group of stable acceptable systems in terms of various constraints on the design and system performance limitations are optimized to achieve the optimum of a performance objective cost [16], [17].

A. Performance Requirements

Commonly, the tracking performance is the main objective of the control system. For this purpose, we assume that a reference unit step is applied at the input of the system in Fig.1 with an initial condition $x(0)=0$. The closed-loop system is asymptotically stable so that $y(t)$ will reach to $v(t)$, i.e., $y(\infty)=v(t)=1$, and $u(t)$ will approach zero as $(t\to\infty)$ [18]. The steady state of the system is described by

$$\begin{aligned}\dot{\bar{x}}(\infty) &= \Lambda\bar{x}(\infty)+\bar{B}K'_{DC}v(\infty)\\ y(\infty) &= \bar{C}\bar{x}(\infty)\end{aligned} \tag{20}$$

By subtracting Eq. (20) from Eq. (5), one get

$$\begin{aligned}\dot{\bar{e}}(t) &= \Lambda\bar{e}(t)\\ e_y(t) &= y(t)-y(\infty)=CE\mathrm{e}^{\Lambda t}\bar{e}(0)\end{aligned}$$

where $\bar{e}(t)=\bar{x}(t)-\bar{x}(\infty)=\mathrm{e}^{\Lambda t}\bar{e}(0)$, $e_y(t)$ is the tracking error, and e is the exponential expression. The value of $\bar{e}(0)$ is computed from Eq. (20) where at the steady state $\dot{\bar{x}}(\infty)=0$,

$$\bar{x}(\infty)=-\Lambda^{-1}\bar{B}K'_{DC},$$

and $\bar{e}(0)=-\bar{x}(\infty)=\Lambda^{-1}\bar{B}K'_{DC}$,
which yields

$$e_y(t)=\bar{E}E_t \tag{21}$$

where $\bar{E}=K'_{DC}CE\Lambda^{-1}$ is an $1\times n$ matrix and $E_t=\mathrm{e}^{\Lambda t}\bar{B}=[\mathrm{e}^{\lambda_1 t}\ \mathrm{e}^{\lambda_2 t}\ \ldots\ \mathrm{e}^{\lambda_n t}]^T$. Indeed, the quantitative measurement of the system tracking performance is the integral of square error (ISE) between reference input and corresponding output which is given as [11]

$$\begin{aligned}\underset{(\tau,\zeta_i,\omega_i)}{J} &= \int_0^\infty e_y(t)^2\,dt=\int_0^\infty \bar{E}\left[E_tE_t^T\right]_{n\times n}\bar{E}^T dt\\ &= \bar{E}\left(\int_0^\infty\begin{bmatrix}\mathrm{e}^{2(\lambda_1)t} & \mathrm{e}^{(\lambda_1+\lambda_2)t} & \ldots & \mathrm{e}^{(\lambda_1+\lambda_n)t}\\ \mathrm{e}^{(\lambda_2+\lambda_1)t} & \mathrm{e}^{2(\lambda_2)t} & \ldots & \mathrm{e}^{(\lambda_2+\lambda_n)t}\\ \vdots & \vdots & \vdots & \vdots\\ \mathrm{e}^{(\lambda_n+\lambda_1)t} & \mathrm{e}^{(\lambda_n+\lambda_2)t} & \ldots & \mathrm{e}^{2(\lambda_n)t}\end{bmatrix}dt\right)\bar{E}^T\\ &= \bar{E}\left[\frac{-1}{\lambda_i+\lambda_j}\right]_{n\times n}\bar{E}^T=\sum_{i=1}^n\sum_{j=1}^n\frac{-\bar{E}_i\bar{E}_j}{\lambda_i+\lambda_j}\end{aligned} \tag{22}$$

The optimal system performance is achieved at the minimum of $J(\tau,\zeta_i,\omega_i)$. However, the minimization of the performance objective $J(\tau,\zeta_i,\omega_i)$ should be carried out under system performance limitations. These limitations will be highlighted as design constraints.

B. Performance Constraints

Practically, the autopilot must be designed in concert with the required frequency response and within the system capabilities. In the following, these performance constraints are manipulated in the form of inequalities and in terms of the design parameters.

1) Frequency domain constraints

Several design specifications and performance limitations can be converted into constraints on the shape of the open-loop gain $|L(j\omega_s)|$. The open-loop transfer function is from the process input u to the controller's output $\hat{u}$ with zero reference input (Fig. 1). The state-space model from u to $\hat{u}$ is given by $\dot{x}(t)=Ax(t)+Bu(t)$ and $\hat{u}=-Kx$ which corresponds to the following open-loop *negative feedback* transfer function

$$L(s)=K(sI-A)^{-1}B=\frac{1}{a(s)}KQS \tag{23}$$

where $a(s)=|sI-A|$, Q and S are defined as in Eq. (14). Substituting by $s=j\omega_s$, the open loop magnitude ratio is written as

$$\begin{aligned}|L(j\omega_s)|^2 &= L(j\omega_s)^*L(j\omega_s)\\ &= \frac{K(QS^*)(QS)^TK^T}{a(-j\omega_s)a(j\omega_s)}\\ &= \frac{B^\dagger\hat{A}(Q\bar{S}_{\omega_s}Q^T)(B^\dagger\hat{A})^T}{\bar{a}(\omega_s)}\end{aligned} \tag{24}$$

where $S^*=[(-j\omega_s)^{n-1}\cdots(-j\omega_s)^2\ -j\omega_s\ 1]^T$, $\bar{S}_{\omega_s}=S^*S^T$ is an $n\times n$ matrix defined by the frequency ω_s, and $\bar{a}(\omega_s)=a(-j\omega_s)a(j\omega_s)$ is a polynomial in ω_s. Eq. (24) provides an exact value of the open-loop gain corresponding to certain gain design $K(\tau,\zeta_i,\omega_j)=B^\dagger\hat{A}$ at any specific frequency ω_s.

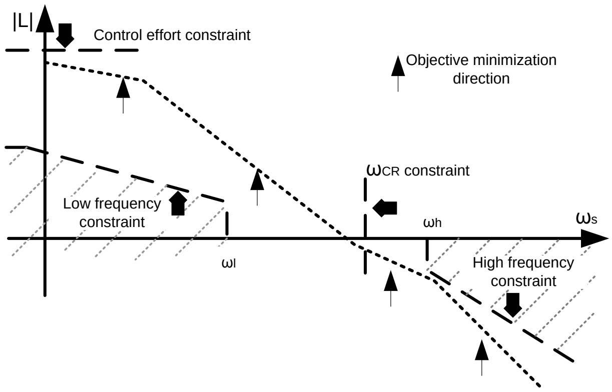


Fig. 2. System open-loop performance.

As noted above, the shape of the open-loop gain could provide important aspects of the closed-loop control system performance. Commonly, this shape is defined by three specifications which are the minimum level of the low frequencies gain, the crossover frequency value, and the high frequencies attenuation [15]. The open-loop performance is illustrated in Fig. 2. In general, the tracking performance and disturbance rejection properties require a wide closed-loop bandwidth, while robustness to un-modeled dynamics requires smaller bandwidth. Namely, this challenge is addressed by limiting the crossover frequency value to make sure that the open-loop gain is below some desired level at high frequencies [19]. The open loop magnitude ratio at the crossover frequency is equal to one, i.e., $|L(j\omega_{CR})|^2 = 1$. Therefore, the first constraint is on the crossover frequency value which is defined by the following inequality

$$\left.\frac{B^{\dagger}\hat{A}(Q\bar{S}_{\omega_s}Q^T)(B^{\dagger}\hat{A})^T}{\bar{a}(\omega_s)}\right|_{\omega_s=\omega_{CRd}} < 1 \tag{25}$$

where ω_{CRd} is a prescribed limitation on ω_{CR}, i.e., for any gain design $K = B^{\dagger}\hat{A}$ that satisfies the inequality in Eq. (25) then the corresponding ω_{CR} will be $\omega_{CR} \le \omega_{CRd}$. For the purpose of missile autopilot design, a "*classical rule of thumb*" limits this value to be less than or equal one third of the actuating natural frequency [20]. The crossover frequency constraint is equivalently expressed as

$$g_{CR}(\tau,\zeta_i,\omega_j) = B^{\dagger}\hat{A}(Q\bar{S}_{\omega_{CRd}}Q^T)(B^{\dagger}\hat{A})^T - \bar{a}(\omega_{CRd}) < 0 \tag{26}$$

Moreover, at high frequencies it is desirable that the $|L(s)|$ to be small so that $|L(s)|/(1+|L(s)|)$ is small; this ensures that the system output will be relatively insensitive to the measurement noise. The required attenuation level for a specified frequency point at the high frequency $\omega_{hi} > \omega_{CRd}$ is constrained by

$$|L(j\omega)|_{\omega=\omega_{hi}} = \sqrt{\frac{B^{\dagger}\hat{A}(Q\bar{S}_{\omega_{hi}}Q^T)(B^{\dagger}\hat{A})^T}{\bar{a}(\omega_{hi})}} < r_{hi} < 1$$

where r_{hi} is a positive fraction corresponding to the –dB attenuation level. The high frequency attenuation level is decided either depending on specified design requirements or based classical design roles. The high frequency constraint is given as

$$g_{hi}(\tau,\zeta_i,\omega_j) = B^{\dagger}\hat{A}(Q\bar{S}_{\omega_{hi}}Q^T)(B^{\dagger}\hat{A})^T - r_{hi}^2\bar{a}(\omega_{hi}) < 0 \tag{27}$$

Similarly, the open loop gain at low frequencies $\omega_{li} < \omega_{CRd}$ is required to be larger than certain level to ensure good command tracking and low sensitivity to plant variations. This level could be specified with positive value $r_{li} > 1$ as

$$g_{li}(\tau,\zeta_i,\omega_j) = \frac{1}{B^{\dagger}\hat{A}(Q\bar{S}_{\omega_{li}}Q^T)(B^{\dagger}\hat{A})^T} - \frac{1}{r_{li}^2\bar{a}(\omega_{li})} < 0 \tag{28}$$

2) Control effort constraints

The hardware limits of the actuating system should be considered; otherwise, the performance of the closed-loop system may be significantly degraded or may even become unstable in the saturation situation [21]. Generally, the saturation problem could be safely ignored, if the autopilot does not command the actuator to exceed its limits. Consequently, the physical limitations of the actuating system should consider as a constraint on the performance of the designed autopilot. To this end, the transfer function between input demand acceleration and autopilot command, Fig. 1, is given by

$$\frac{u(s)}{v(s)} = K'_{DC}\frac{1}{1+L(s)} = \left(\frac{p_0}{Q_0}\right)\frac{1}{1+L(s)} \tag{29}$$

The maximum demanded control effort $u_{\max}$ corresponding to a specified maximum input demand $v_{\max}$ is defined by

$$u_{\max} = \bar{\sigma}\left(\frac{1}{1+L(s)}\right)\left(\frac{p_0}{Q_0}\right)v_{\max} \tag{30}$$

where $\bar{\sigma}(\bullet)$ denotes the maximum singular value. The maximum control effort should be limited referring to the physical limitation of the actuating system $u_{\lim}$, i.e., $u_{\max} < u_{\lim}$. Then Eq. (30) could be reformed as

$$\bar{\sigma}\left(\frac{1}{1+L(s)}\right)\left(\frac{p_0}{Q_0}\right)v_{\max} < u_{\lim}$$

Equivalent, it could be rearranged as

$$Q_0\left(\frac{u_{\lim}}{v_{\max}}\right)\underline{\sigma}\left(1+L(s)\right) > p_0 > 0$$

where $\underline{\sigma}\left(1+L(s)\right)$ is the minimum singular value of $\left(1+L(s)\right)$. This value reaches to one where $L(\infty) \cong 0$. In the same meaning, the design parameters in p_0 are constrained by

$$\Delta_{(\tau,\omega_i)} = \left(\tau^{-1}\right)^l\left(\prod_{i=1}^{(n-l)/2}\omega_i^2\right) - Q_0\left(\frac{u_{\lim}}{v_{\max}}\right) < 0 \tag{31}$$

where $l = 1$ for odd n and $l = 0$ for even n. Since the characteristic polynomial of the closed-loop could be defined by $1 + L(s) = 0$, the limit on the p_0 value is in turn limiting the open-loop gain at $s = 0$ where

$\lim_{s\to 0}|sI - \Lambda| = p_0$.

C. Constrained Optimization

It is reasonable now, based on the previous discussion, to state the optimal autopilot design problem in the form of inequality constrained optimization problem as

$$\begin{array}{lll} \min\limits_{\tau,\zeta_i,\omega_j} & : & J(\tau,\zeta_i,\omega_j) \\ \text{Subject to} & : & \begin{array}{l} g_{CR}, g_{hi}, g_{li} < 0, \\ {}_{(\tau,\zeta_i,\omega_j)} \\ \Delta_{(\tau,\omega_j)} < 0 \end{array} \\ \text{Bounds} & : & \begin{array}{l} \tau > 0, \omega_i > 0, \\ 1 > \zeta_i > \zeta_{\min} \end{array} \end{array} \quad (32)$$

The target of the optimization problem (32) is to minimize the objective function while satisfying the performance constraints within applicable bounds of the design parameters. Moreover, the constraints divide the optimization space into two domains, the feasible domain where the constraints are satisfied, and the infeasible domain where at least one of the constraints is violated. Mostly, the optimum design is found on the boundary between the feasible and infeasible domains, that is at a point where at least one of the constraints equal zero. The method of Lagrange Multipliers could apply to solve the optimization problem. Particularly, this problem is a nonlinear constrained multi-variable optimization problem. As expected, the function fmincon of the MATLAB Optimization Toolbox can solve such kind of smooth objective optimization problem well effectively with feasible initial design parameters. Moreover, it converges to the same minimum point even starting from different initial guesses [9]. In this line of thought, the optimal autopilot gain K is easily calculated by substituting the optimum parameters τ, ζ_i and ω_j into Eq. (16).

IV. Numerical Simulation

The proposed method is applied to design a three-loop longitudinal missile autopilot. The system model is describe as in Eq. (1) where

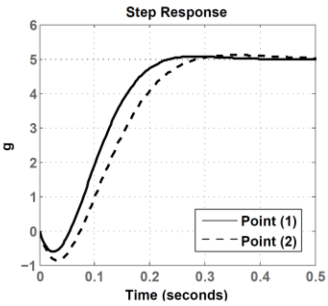
$$A = \begin{bmatrix} -Z_\alpha & VZ_\alpha & 0 \\ 0 & 0 & 1 \\ M_\alpha / V & -M_\alpha & 0 \end{bmatrix}, \; B = \begin{bmatrix} VZ_\delta \\ 0 \\ -M_\delta \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

$x(t) = [a_y \; q \; \dot{q}]^T$, q is the pitch angular rate, a_y is the missile acceleration, V is the missile velocity and M_α, M_δ, M_q, Z_α and Z_δ are the aerodynamics coefficients. The system parameters are given in Table I for two different flight conditions with same speed $V = 914$ m/s and different attitude H [22]. Moreover, the actuator is considered to be a second-order dynamic system with natural frequency $\omega_{ACT} = 220$ rad/s, damping factor $\zeta_{ACT} = 0.65$, maximum fin deflection rate $u_{\lim} = 300$ deg/s and maximum fin deflection ± 10 deg.

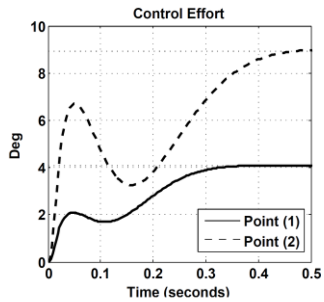
TABLE I: Typical Missile Aerodynamic Data

Point no.	H m	M_α s^{-2}	M_δ s^{-2}	Z_α s^{-1}	Z_δ s^{-1}
(1)	9150	240	204	1.17	0.239
(2)	15250	99.1	81.7	0.533	0.0957

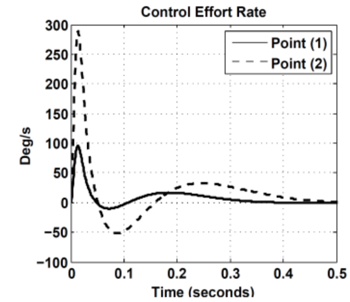
The proposed approach is used in to provide the optimal autopilot gain K where the control signal $u(t)$ is referring to the demanded fin deflection rate. The autopilot system is required to track maximum commanded acceleration $a_{yc_{\max}} = 5g$, with a steady state error of no more than one percent and a time constant of less than 0.25sec. The open-loop crossover frequency is limited by 50 rad/s, the system open loop gain required to be greater than 3dB at 35Hz and less than −15dB at 300Hz. Moreover, the designed system should satisfy these requirements at the two operating and avoid the actuator saturation problems. The design performance is tested for an acceleration command of 5g. The corresponding tracking performances, fin deflection and fin deflection rate are exhibited in Fig. 3 and the open-loop gain is shown in Fig. 4. The numerical results are stated in Table II for theses two operating conditions.



(a) Response of 5g acceleration



(b) Fin-defection response



(c) Fin-defection rate response

Fig. 3. Design performance against 5g input command.

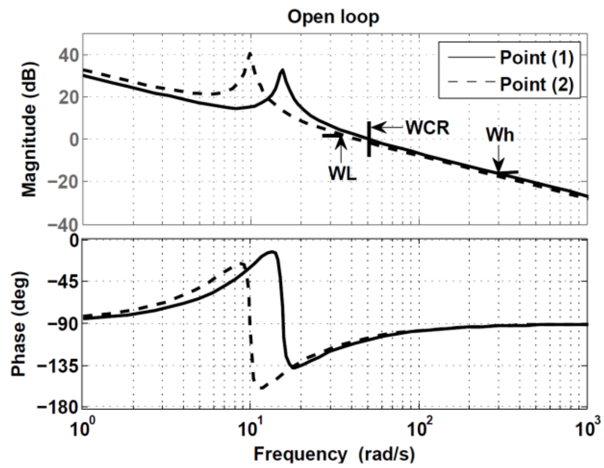


Fig. 4. Open-loop frequency response.

As stated in Table I, there is a large difference between the dynamic parameters of the system at the two operating conditions. Consequently, the design process using available optimal technique such as linear quadratic regulator may requires several trials of weights tuning in order to provide the system with the required performance. In contrast, the proposed approach handles the design based on explicit forms of the objective function and performance constraints, which are same at any operating point. The only change between certain point and another is the values of the system parameters. In other words, the designer is no longer need to adjust design parameters or weightings at each operating condition to reach the desired performance. In this case, the optimal design at any operating point is achieved by solving the optimization problem in Eq. (32). Moreover, the simulation results show that the proposed method exhibits good tracking performance within the design requirements and system limits for the two operating points. Moreover, as stated in Table II, these performances are achieved with phase margin greater than 70^o which indicates a high robustness level. In other word, the performance, the stability and the robustness of the autopilot system are provided with this control technique. Moreover, these results are achieved without the burden of weightings adjustment.

TABLE II: SIMULATION RESULTS

Specifications	Point (1)	Point (2)
Optimal Design Parameters	$\tau = 0.058$ $\zeta = 0.7$ $\omega = 21.4$	$\tau = 0.06$ $\zeta = 0.76$ $\omega = 16$
Optimal Gain	$K_1 = 0.044$ $K_2 = 3.347$ $K_3 = 0.272$	$K_1 = 0.13$ $K_2 = 6.68$ $K_3 = 0.638$
Rise Time (s)	0.126	0.15
Settling Time (s)	0.2175	0.265
Over Shoot %	1.4	2.19
Phase Margin (°)	72.6	71.75
ω_{CR} (rad/s)	50	43

V. CONCLUSIONS

An optimal state feedback design technique is proposed and applied successfully to design a three-loop missile autopilot system. The design practicality and robustness are provided by considering both the system frequency response and the actuating system limits. Moreover, this method allows the designer to apply classical loop-shaping concepts to obtain good performance while optimizing the response near the system bandwidth to achieve robust stabilization. The optimal design is achieved using constrained optimization technique for the optimum of the tracking performance. The whole design is established in terms of the stable parameters of the desired closed-loop diagonal form and in such way that the tradeoff between system performance and robustness is brought directly into the design process without required for weightings parameters.

APPENDIX

Proof of Theorem 1.

Let A, B are two matrices that are defined as described in Eq. (1) where

$$\tilde{q}(s)B = \mathrm{adj}(sI - A)B = \tilde{q}_{n-1}Bs^{n-1} + \cdots + \tilde{q}_1 Bs + \tilde{q}_0 B$$

$$q(s)B = \mathrm{adj}(sI - A + BK)B = q_{n-1}Bs^{n-1} + \cdots + q_1 Bs + q_0 B$$

$\tilde{q}_0, \ldots, \tilde{q}_{n-1}$ and $q_0, \ldots, q_{n-1}$ are constant $n \times n$ matrices. The two terms $\tilde{q}(s)B$ and $q(s)B$ are equivalent if $\tilde{q}_i = q_i$ where $i = 0,1\ldots,n-1$. For $i = n-1$, using Leverrier-Faddeev method, it is found that $\tilde{q}_{n-1} = q_{n-1} = I$ then $\tilde{q}_{n-1}B = q_{n-1}B$.

Similarly, for $i = n-2$, it can be shown that

$$\tilde{q}_{n-2} = A\tilde{q}_{n-1} - \mathrm{tr}(A)I_n \qquad q_{n-2} = (A+BK)\tilde{q}_{n-1} - \mathrm{tr}(A+BK)I_n$$

$$\tilde{q}_{n-2} = A - \mathrm{tr}(A) \qquad q_{n-2} = A + BK - \mathrm{tr}(A)I_n - \mathrm{tr}(BK)I_n$$

$$\tilde{q}_{n-2}B = AB - \mathrm{tr}(A)B \qquad q_{n-2}B = AB + \underline{BKB} - \mathrm{tr}(A)B - \underline{\mathrm{tr}(BK)B}$$

As $BKB = \mathrm{tr}(BK)B$, then the term $q_{n-2}B$ is reduced to $q_{n-2}B = AB - \mathrm{tr}(A)B$, i.e., $\tilde{q}_{n-2}B = q_{n-2}B$. For $i = n-3$, in the same way, one can obtain

$$\tilde{q}_{n-3} = A^2 - \mathrm{tr}(A)A - \frac{1}{2}\left(\mathrm{tr}(A^2) - \mathrm{tr}(\mathrm{tr}(A)A)\right)I_n$$

$$\tilde{q}_{n-3}B = A^2B - \mathrm{tr}(A)AB - \frac{1}{2}\left(\mathrm{tr}(A^2) - \mathrm{tr}(\mathrm{tr}(A)A)\right)B$$

$$\begin{aligned} q_{n-3}B = {} & A^2B + \underline{BKAB} + \underline{ABKB} + \underline{BKBKB} - \mathrm{tr}(A)AB - \underline{\mathrm{tr}(A)BKB} \cdots \\ & - \underline{\mathrm{tr}(BK)AB} - \underline{\mathrm{tr}(BK)BKB} - \frac{1}{2}\left(\mathrm{tr}(A^2) - \mathrm{tr}(\mathrm{tr}(A)A)\right)B \cdots \\ & - \frac{1}{2}[\underline{\mathrm{tr}(BKA)} + \underline{\mathrm{tr}(ABK)} + \underline{\mathrm{tr}(BKBK)} - \underline{\mathrm{tr}(\mathrm{tr}(A)BK)} \cdots \\ & - \underline{\mathrm{tr}(\mathrm{tr}(BK)A)} - \underline{\mathrm{tr}(\mathrm{tr}(BK)BK)}]B \end{aligned}$$

As can be seen the term $\tilde{q}_{n-3}B = q_{n-3}B$ where

$$BKAB = \mathrm{tr}(BKA)B = \mathrm{tr}(ABK)B,$$

$$ABKB = \mathrm{tr}(BK)AB, \; BKBKB = \mathrm{tr}(BK)BKB,$$

$$\mathrm{tr}(A)BKB = \mathrm{tr}(\mathrm{tr}(BK)A)B = \mathrm{tr}(\mathrm{tr}(A)BK)B,$$

and $\mathrm{tr}(BKBK)B = \mathrm{tr}(\mathrm{tr}(BK)BK)B$. It also can be notice that all terms content K are vanished; this is hold for $\tilde{q}_i = q_i$. Moreover, this result is consistent with the fact that the closed-loop and the original plant of the system described in Fig. 1 share the same nominator. Finally, it can be concluded that $adj(sI - A)B = adj(sI - A + BK)B$.

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