Positive Solutions for a Kind of Multipoint Boundary Value Problem

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***Abstract*—In this paper, several existence results of positive solutions are obtained for a class of multi-point boundary value problems with p -Laplacian. By applying a monotone iterative method, not only we obtain the existence of positive solutions for the problem, but also establish the corresponding iterative schemes.**

***Index Terms*—Successive iteration, positive solutions, boundary value problem, p -Laplacian.**

I. Introduction

In this paper, we will consider the positive solutions to the following multi-point boundary value problem with p -Laplacian

$$(\phi_p(u'(t)))' + f(t,u(t)) = 0, \qquad t \in (0,1), \tag{1}$$

$$u'(0) = \sum_{i=1}^{n-2} \alpha_i u'(\xi_i),\ u(1) = \sum_{i=1}^{n-2} \beta_i u(\xi_i), \tag{2}$$

where $\phi_p(s) = |s|^{p-2} s$, $p > 1$, $\xi_i \in (0,1)$

with $0 < \xi_1 < \xi_2 < \cdots < \xi_{n-2} < 1$,

and α_i, β_i, f satisfy

$(H1)$: $0 \le \alpha_i, \beta_i < 1\,(i = 1,2,\ldots,n-2)$ satisfy

$0 \le \sum_{i=1}^{n-2} \alpha_i < 1$, $0 < \sum_{i=1}^{n-2} \beta_i < 1$;

$(H2)$: $f(t,u) \in C([0,1] \times [0,+\infty) \to [0,+\infty))$.

The study of positive solutions on second-order boundary value problems for ordinary differential equations has aroused extensive interest, one may see [1]-[5] and references therein.

Among the substantial number of works dealing with nonlinear differential equations we mention the boundary value problem (1) and (2). One thing to be mentioned is that the monotone iteration scheme is an interesting and effective technique for investigating the existence of solutions of nonlinear problems.

However, most of papers studied the existence of the positive solutions of various boundary value problems by various fixed point theorems, one may see [1]-[4], [6], there are only a few papers which concern with the computational methods of boundary value problems, for example in [5], [7] . Then, it comes to us a question that "How can we find the solutions when they are known to exist?" Motivated by this question, and what mentioned above, the aim of this paper is to establish some simple criteria for the iteration and existence of positive solutions for the problem (1) and (2).

We emphasize that the construction of the monotone iterative schemes in our work does not require the existence of lower and upper solutions for the problems we will study and it starts off with known simple functions. It is worth stating that our work give a way to find the solutions which will be useful from an application viewpoint.

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II. Preliminaries

In this section, we present here two generally definitions from cone theory.

Definition 1. Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \ge 0$, $b \ge 0$ and

(ii) $u, -u \in P$ imply $u = 0$.

Definition 2. The map α is said to be concave on [0,1], if

$$\alpha(tu + (1-t)v) \ge t\alpha(u) + (1-t)\alpha(v)$$

for all $u, v \in [0,1]$ and $t \in [0,1]$.

Let the Banach space $E = C[0,1]$ be endowed with the norm

$$\|u\| := \max_{0 \le t \le 1} |u(t)|.$$

We denote

$$E_+ = C_+[0,1] = \{u \in E \mid u(t) \ge 0, t \in [0,1]\},$$

and define the cone $P \subset E$ by $P = \{u \in E \mid u(t) \ge 0, u \text{ is concave and nonincreasing on } [0,1]\}$.

Throughout, it is assumed that $(H1)$ and $(H2)$ hold.

Lemma 1. Suppose $y \in C^1[0,1]$ with $(\phi_p(y'(t)))' \in C^1[0,1]$ satisfies

$$-(\phi_p(y'(t)))' \ge 0, \qquad t \in (0,1),$$

$$y'(0) = \sum_{i=1}^{n-2} \alpha_i y'(\xi_i), \quad y(1) = \sum_{i=1}^{n-2} \beta_i y(\xi_i).$$

Then, $y(t)$ is concave and $y(t) \ge 0$, i.e., $y \in P$, and $y'(t) \le 0$ on $[0,1]$.

For any $x \in C_+[0,1]$, suppose u is a solution of the problem

$$\begin{cases}(\phi_p(u'))' + f(t,x) = 0, \ 0 \le t \le 1, \\ u'(0) = \sum_{i=1}^{n-2} \alpha_i u'(\xi_i), \ u(1) = \sum_{i=1}^{n-2} \beta_i u(\xi_i), \end{cases}$$

then we have

$$u'(t) = \phi_p^{-1}\left(W_x - \int_0^t f(s, x(s))ds\right),$$

$$u(t) = -\frac{\sum_{i=1}^{n-2}\beta_i \int_{\xi_i}^{1} \phi_p^{-1}\left(W_x - \int_0^s f(r, x(r))dr\right)ds}{1-\sum_{i=1}^{n-2}\beta_i} - \int_t^1 \phi_p^{-1}\left(W_x - \int_0^s f(r, x(r))dr\right)ds,$$

where W_x satisfy

$$\phi_p^{-1}(W_x) = \sum_{i=1}^{n-2}\alpha_i \phi_p^{-1}\left(W_x - \int_0^{\xi_i} f(r, x(r))dr\right). \tag{3}$$

Lemma 2. For any $x \in C_+^1[0,1]$, there exists a unique W_x satisfies (3) and

$$W_x \in [-\frac{\phi_p(\sum_{i=1}^{n-2}\alpha_i)}{1-\phi_p(\sum_{i=1}^{n-2}\alpha_i)}\int_0^1 f(r,x(r))dr, 0].$$

Proof. For any $x \in C_+^1[0,1]$, define

$$H_x(c) = \phi_p^{-1}(c) - \sum_{i=1}^{n-2}\alpha_i \phi_p^{-1}\left(c - \int_0^{\xi_i} f(r, x(r))dr\right)$$

then $H_x(c) \in C((-\infty,+\infty), R)$ and $H_x(0) \le 0$.

In what follows, we will prove $H_x(c) = 0$ has a unique solution on $(-\infty,+\infty)$, which means there exists a unique $W_x \in (-\infty,+\infty)$ satisfying (3) firstly.

Case i. If $H_x(0) = 0$, then

$$\sum_{i=1}^{n-2}\alpha_i \phi_p^{-1}\left(\int_0^{\xi_i} f(r, x(r))dr\right) = 0$$

$$\Rightarrow \alpha_i \phi_p^{-1}\left(\int_0^{\xi_i} f(r, x(r))dr\right) = 0,$$

for $i = 1, 2, \ldots n-2$.

So
$$\begin{aligned} H_x(c) &= \phi_p^{-1}(c) - \sum_{i=1}^{n-2}\alpha_i \phi_p^{-1}\left(c - \int_0^{\xi_i} f(r, x(r))dr\right) \\ &= \phi_p^{-1}(c) - \sum_{i=1}^{n-2} \phi_p^{-1}\left(\phi_p(\alpha_i)c - \phi_p(\alpha_i)\int_0^{\xi_i} f(r, x(r))dr\right) \\ &= \phi_p^{-1}(c) - \sum_{i=1}^{n-2}\alpha_i \phi_p^{-1}(c) \\ &= (1-\sum_{i=1}^{n-2}\alpha_i)\phi_p^{-1}(c). \end{aligned}$$

Obviously, there exists a unique $c = 0$ satisfying $H_x(c) = 0$. So in this case, this lemma is proved.

Case ii. $H_x(0) \ne 0$, then $H_x(0) > 0$.

1) When $c \in (0,+\infty)$,

$$\begin{aligned} H_x(c) &= \phi_p^{-1}(c) - \sum_{i=1}^{n-2}\alpha_i \phi_p^{-1}\left(c - \int_0^{\xi_i} f(r, x(r))dr\right) \\ &\ge \phi_p^{-1}(c) - \sum_{i=1}^{n-2}\alpha_i \phi_p^{-1}(c) \\ &= (1-\sum_{i=1}^{n-2}\alpha_i)\phi_p^{-1}(c) \\ &> 0. \end{aligned}$$

2) When $c \in (-\infty, 0)$,

$$\begin{aligned} H_x(c) &= \phi_p^{-1}(c) - \sum_{i=1}^{n-2}\alpha_i \phi_p^{-1}\left(c - \int_0^{\xi_i} f(r, x(r))dr\right) \\ &\ge \phi_p^{-1}(c)\left(1 - \sum_{i=1}^{n-2}\alpha_i \phi_p^{-1}\left(1 - \frac{\int_0^{\xi_i} f(r, x(r))dr}{c}\right)\right). \end{aligned}$$

Denote

$$\overline{H(c)} = 1 - \sum_{i=1}^{n-2}\alpha_i \phi_p^{-1}\left(1 - \frac{\int_0^{\xi_i} f(r, x(r))dr}{c}\right)$$

then we obtain that

$$H_x(c) = \phi_p^{-1}(c)\overline{H(c)}.$$

Obviously, $\overline{H(c)}$ is strictly decreasing on $(0,+\infty)$.

Let
$$\bar{c} = \frac{\phi_p(\sum_{i=1}^{n-2}\alpha_i)}{1-\phi_p(\sum_{i=1}^{n-2}\alpha_i)}\int_0^1 f(r, x(r))dr,$$

then $\bar{c} < 0$, and we have $\overline{H(\bar{c})} \ge 0$.

So

$$H_x(\bar{c}) = \phi_p^{-1}(\bar{c})\overline{H(\bar{c})} \le 0.$$

To take into account $H_x(0) > 0$, by the mean value theorem we obtain that there exists a $c_0 \in [\bar{c}, 0) \subset (-\infty, 0)$, such that $H_x(c_0) = 0$.

If there are two constants $c_i \in (-\infty,0) (i = 1, 2)$ satisfying $H_x(c_1) = H_x(c_2) = 0$, then $\overline{H(c_1)} = \overline{H(c_2)} = 0$, so $c_1 = c_2$ since $\overline{H(c)}$ is strictly decreasing on $(-\infty, 0)$.

Therefore, $H_x(c) = 0$ has a unique solution on $(-\infty, 0)$. Combining those cases above, we obtain $H_x(c) = 0$ has a unique solution on $[\bar{c}, 0]$, which means there exists a unique $W_x \in [\bar{c}, 0]$. The proof is completed.

III. Main Results

For notational convenience, we denote

$$A = \frac{(1-\sum_{i=1}^{n-2}\beta_i\xi_i)}{(1-\sum_{i=1}^{n-2}\beta_i)\phi_p^{-1}\left(1-\phi_p(\sum_{i=1}^{n-2}\alpha_i)\right)},$$

$$B = \frac{p-1}{p}\frac{1-\sum_{i=1}^{n-2}\beta_i\xi_i^{\frac{p}{p-1}}}{1-\sum_{i=1}^{n-2}\beta_i}.$$

Theorem 1. Assume that (H1) and (H2) hold, and there exist $0 < b < a$, such that

$(H3)$: $f(t, x_1) \le f(t, x_2)$ for any $0 \le t \le 1, 0 \le x_1 \le x_2 \le a$;

$(H4)$: $\max_{0\le t\le 1} f(t, a) \le \phi_p(\frac{a}{A})$;

$(H5)$: $\min\limits_{0\le t\le 1} f(t,0) \ge \phi_p(\frac{b}{B})$;

Then the boundary value problem (1) and (2) has at least one positive, concave and nonincreasing solution w^* or v^*, such that

$$b \le \|w^*\| \le a, \quad and \quad \lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0 = w^*$$

where $w_0(t) = a, \quad 0 \le t \le 1,$

and
$$b \le \|v^*\| \le a, \quad and \quad \lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0 = v^*,$$

where $v_0(t) = b(1-t), \ 0 \le t \le 1,$

where

$$(Tu)(t) = -\frac{\sum_{i=1}^{n-2}\beta_i \int_{\xi_i}^{1} \phi_p^{-1}\Big(W_x - \int_0^s f(r,x(r))dr\Big)ds}{1-\sum_{i=1}^{n-2}\beta_i} - \int_t^1 \phi_p^{-1}\Big(W_x - \int_0^s f(r,x(r))dr\Big)ds. \tag{4}$$

The iterative scheme in Theorem 1 is

$$w_0(t) = a,$$
$$w_{n+1} = Tw_n = T^n w_0, \ n = 0,1,2\ldots$$

which starts off with a simple known constant function or

$$v_0(t) = b(1-t), v_{n+1} = Tv_n = T^n v_0, \ n = 0,1,2\ldots$$

which starts off with a simple known linear function.

Proof. We define an operator $T: P \to E$ by (4), then it is easy to obtain that, for each $u \in P$, there is $Tu \in C^1[0,1]$ is nonnegative and satisfies (2). Moreover, by Lemma 1 we have, Tu is concave. So, $T: P \to P$.

And a standard argument shows that $T: P \to P$ is completely continuous, and each fixed point of T in P is a solution of problem (1) and (2). We can also obtain that T is nondecreasing about u on $[0,+\infty)$.

We denote

$$P[b,a] = \{u \in P \mid b \le \|u\| \le a\}.$$

We now firstly show that $T: P[b,a] \to P[b,a]$.

If $u \in P[b,a]$, then by Lemma 1, we have $0 \le u(t) \le \max\limits_{0\le t\le 1} u(t) = u(0) = \|u\| \le a.$

Then, by $(H3)-(H5)$,

$$\begin{aligned} 0 &\le f(t,u(t)) \le f(t,a) \\ &\le \max_{0\le t\le 1} f(t,a) \le \phi_p(\frac{a}{A}), \quad \text{for } 0\le t\le 1. \\ f(t,u(t)) &\ge f(t,0) \\ &\ge \min_{0\le t\le 1} f(t,0) \ge \phi_p(\frac{b}{B}), \quad \text{for } 0\le t\le 1. \end{aligned}$$

Moreover, by lemma 2 we have

$$\begin{aligned} \|Tu\| &= \max\nolimits_{0\le t\le 1} |(Tu)(t)| \\ &= Tu(0) \\ &\le \frac{\sum_{i=1}^{n-2}\beta_i \int_{\xi_i}^{1} \phi_p^{-1}\Big(\frac{\phi_p(\sum_{i=1}^{n-2}\alpha_i)}{1-\phi_p(\sum_{i=1}^{n-2}\alpha_i)}\int_0^1 f(r,x(r))dr + \int_0^s f(r,x(r))dr\Big)ds}{1-\sum_{i=1}^{n-2}\beta_i} \\ &\quad + \int_0^1 \phi_p^{-1}\Big(\frac{\phi_p(\sum_{i=1}^{n-2}\alpha_i)}{1-\phi_p(\sum_{i=1}^{n-2}\alpha_i)}\int_0^1 f(r,x(r))dr + \int_0^s f(r,x(r))dr\Big)ds \\ &\le \frac{a}{A}\frac{(1-\sum_{i=1}^{n-2}\beta_i\xi_i)}{(1-\sum_{i=1}^{n-2}\beta_i)\phi_p^{-1}\Big(1-\phi_p(\sum_{i=1}^{n-2}\alpha_i)\Big)} \\ &= a. \end{aligned}$$

and

$$\begin{aligned} \|Tu\| &= \max\nolimits_{0\le t\le 1} |(Tu)(t)| \\ &= Tu(0) \\ &\ge \frac{\sum_{i=1}^{n-2}\beta_i \int_{\xi_i}^{1} \phi_p^{-1}\Big(\int_0^s f(r,x(r))dr\Big)ds}{1-\sum_{i=1}^{n-2}\beta_i} \\ &\quad + \int_0^1 \phi_p^{-1}\Big(\int_0^s f(r,x(r))dr\Big)ds \\ &\ge \frac{b}{B}\frac{p-1}{p}\frac{1-\sum_{i=1}^{n-2}\beta_i\xi_i^{\frac{p}{p-1}}}{1-\sum_{i=1}^{n-2}\beta_i} = b. \end{aligned}$$

So, $b \le \|Tu\| \le a.$

Thus we get

$$b \le \|Tu\| \le a,$$

which means

$$T: P[b,a] \to P[b,a].$$

Next, we will establish iterative scheme for approximating the solution.

Let

$$w_0(t) = a, 0 \le t \le 1,$$

then

$$w_0(t) \in P[b,a].$$

We denote

$$w_{n+1} = Tw_n = T^n w_0, \ n = 0,1,2\ldots$$

Since $T: P[b,a] \to P[b,a]$, we have $w_n \in TP[b,a] \subseteq P[b,a], \ n = 1,2,\ldots$

Since T is completely continuous, we assert that $\{w_n\}_{n=0}^{\infty}$ is a sequentially compact set and it has convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$, then there exists $w^* \in P[b,a]$ such that $w_{n_k} \to w^*$.

Because

$$0 \le w_1(t) = Tw_0 \le \|Tw_0(t)\| \le a = w_0(t), \quad 0 \le t \le 1,$$

Moreover

$$w_2(t) = Tw_1(t) \le Tw_0(t) = w_1(t), \quad 0 \le t \le 1.$$

Hence, by the induction, then $w_{n+1}(t) \le w_n(t), \quad 0 \le t \le 1, \quad n = 0,1,2\ldots$

Thus, there exists $w^* \in P[a,b]$ such that $w_n \to w^*$. Applying the continuity of T and $w_{n+1} = Tw_n$, we get $Tw^* = w^*$.

Another way to approach this is to start off with a simple linear function.

Let

$$v_0(t) = b(1-t), 0 \le t \le 1,$$

Then

$$v_0(t) \in P[a,b].$$

We denote

$$v_{n+1} = Tv_n = T^n v_0, n = 0,1,2\ldots$$

It is similar to the earlier arguments, we assert that $\{v_n\}_{n=0}^{\infty}$ has convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$, and there exists $v^* \in P[a,b]$ such that $v_{n_k} \to v^*$.

Because

$$\begin{aligned} v_1(t) = Tv_0 &\ge (1-t)\|Tv_0(t)\| \\ &\ge b(1-t) = v_0(t), \quad 0 \le t \le 1, \end{aligned}$$

Moreover

$$v_2(t) = Tv_1(t) \ge Tv_0(t) = v_1(t), \quad 0 \le t \le 1.$$

By an induction argument similar to the above we obtain

$$v_{n+1}(t) \ge v_n(t), \quad 0 \le t \le 1, \quad n = 0,1,2\ldots$$

Thus, there exists $v^* \in P[a,b]$ such that $v_n \to v^*$. Applying the continuity of T and $v_{n+1} = Tv_n$, we get $Tv^* = v^*$.

It is well known that each fixed point of T in P is a solution of problem (1) and (2). Hence, we assert that w^* or v^* is a positive, nonincreasing and concave solutions of the problem (1) and (2).

Furthermore, if $\lim_{n\to\infty} w_n \ne \lim_{n\to\infty} v_n$, then w^* and v^* are two positive, nonincreasing and concave solutions of the problem (1) and (2).

And if $\lim_{n\to\infty} w_n = \lim_{n\to\infty} v_n$,then $w^* = v^*$ is a positive, nonincreasing and concave solution of the problem (1) and (2).

Anyway, the problem (1) and (2) has at least one positive concave solution.

The proof is completed.

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