Estimate on the Risk of Epidemic Outbreaks for an Insurer: A Simple Stochastic Model

Chiaki Ishii, Naoyuki Ishimura, and Takahiko Fujita

***Abstract*—Epidemics bursts have been sometimes observed in recent years, whose examples include, SARS (Severe Acute Respiratory Syndrome) and Ebola virus disease, and so on. For global companies which insure these epidemics, it is important and necessary to estimate the effect of events. Here we introduce a simple stochastic model for pricing such a kind of risks, which involves the Kermack-Mckendrick epidemic model combined with a stochastic trigger variable. The computations of our model are also given.**

***Index Terms*—Epidemic outbreaks, risk for an insurer, stochastic process, threshold theorem.**

I. Introduction

Epidemics outbreaks have been sometimes observed in recent years, whose examples include, to name a few, SARS (Severe Acute Respiratory Syndrome) and Ebola virus disease, and others. Such phenomena are really a challenging issue for modern society.

In order to manage the risk originating from these epidemics, it is without doubt that the first important step should be based on scientific researches. Apart from this aspect, insurance companies, which provide the insurance of epidemics, would want to estimate the extent of the risk. One of positive functions of insurance, in its own nature, certainly stabilize and mitigate the influence of tragedy events.

Here we introduce a simple stochastic model for estimating the above mentioned epidemics bursts.

Mathematical modelling of diseases has been developed and investigated over 300 years. Much progress has been made from various points of view; empirical approaches, deterministic models, stochastic models, and so on. In the present article, we employ the classical deterministic model due to Kermack-Mckendrick, whose important conclusion is the so-called threshold theorem; it becomes a benchmark for later researches. By this, we can tell the criterion that a major outbreaks occurs, and furthermore, obtain a rough estimate of ultimate numbers of infected and removed.

Our idea is that we combine a stochastic process with this epidemic model. The stochastic process is intended to model the trigger variable of the occurrence of epidemic outbreaks, which is somewhat similar to the modelling of catastrophic events such as a big earthquake, a typhoon disaster, an eruption of volcanos, and so on (observe [1] for instance). From the financial standpoints, similarity between catastrophic events and epidemic outbreaks is very strong. As is well known, in catastrophic events, several financial products have been already invented to mitigate the relevant disaster. For instance, we recall catastrophic options. Our wish is to use some of its establishments. In addition to this, a market risk should be also considered. In anyway, we are then able to estimate the risk of the considered epidemics bursts by taking the expectation, which an insurer may find an interest.

The organization of the paper is as follows: In Section II, basic issues of our classical epidemic model as well as stochastic process are recalled. Our model for the estimation of risk is explained in Section III. Section IV is devoted to the computation of risk. Section V concludes with discussion.

II. Preliminary

A. Epidemic Model

We first recall the classical deterministic epidemic model of Kermack-Mckendrick. For background issues and other details, see for instance, nice monographs by Dalley and Gani [2], Murray [3].

Let $x(t)$ denote the number of susceptibles, $y(t)$ infectives, $z(t)$ removals, which are counted according to the disease status. Here removals mean dead, isolated, or immune individuals. The total size of the population $N = x(t) + y(t) + z(t)$ is assumed to be fixed for all $t \geq 0$.

The system of ordinary differential equations due to Kermack-Mckendrick is then expresses as follows:

$$\frac{dx(t)}{dt} = -\beta x(t)y(t) \quad (1)$$

$$\frac{dy(t)}{dt} = \beta x(t)y(t) - \gamma y(t) \quad (2)$$

$$\frac{dz(t)}{dt} = \gamma y(t). \quad (3)$$

The equations is subject to the initial conditions $\big(x(0), y(0), z(0)\big) = (x_0, y_0, 0)$ with $x_0 + y_0 = N$. Here, β denotes the infection parameter representing the strength of epidemics, and γ is the removal parameter indicating the rate of infectives becoming immune. We define the critical parameter $\rho = \gamma/\beta$ as the relative removal rate. It is easy to see that $x(t)$ is monotone non-increasing, and $z(t)$ is non-decreasing. If $x_0 \leq \rho$, then $y(t)$ is monotone decreasing for all $t > 0$.

Now the well-known Kermack-Mckendrick threshold

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theorem is summarized as follows, whose way of formulation is essentially taking from [2]. For further details and other surrounding information, we refer to a comprehensive book of Murray [3].

Theorem (Kermack-Mckendrick). (i) Let $x_\infty = \lim_{t\to\infty} x(t)$ and $z_\infty = \lim_{t\to\infty} z(t)$. Then, when infection ultimately ceases spreading, it follows that

$$N - z_\infty = x_0 + y_0 - z_\infty = x_0 e^{-\frac{z_\infty}{\rho}} \tag{4}$$

where $y_0 < z_\infty < x_0 + y_0 = N$.

1) A major outbreak occurs if and only if $x_0 > \rho$.
2) If $x_0 = \rho + \nu$ with small $\nu > 0$ and $y_0 x$ is small relative to ν, then the total number of susceptibles left in the population and z_∞ are approximately $\rho - \nu$ and 2ν, respectively.

Here we just exhibit a sketch of proof, since the theorem itself is well-known. For the detailed proof, we refer to before mentioned [2], [3].

First we note that

$$\frac{d}{dt}\big(x(t) + y(t) + z(t)\big) = 0 \tag{5}$$

And hence

$$x(t) + y(t) + z(t) = N \tag{6}$$

For all $t \geq 0$.

Next we divide the Eq. (1) by (3) to obtain

$$\frac{dx}{dz} = -\frac{\beta}{\gamma}x = -\frac{1}{\rho}x \tag{7}$$

where $\rho = \gamma/\beta$ denotes the relative removal rate. Integration directly gives

$$x(t) = x_0 e^{-\frac{z(t)}{\rho}} \tag{8}$$

In a similar way we find that

$$\frac{dy}{dx} = -1 + \frac{\rho}{x} \tag{9}$$

And so that

$$x(t) + y(t) - \rho \log x(t) = x_0 + y_0 - \rho \log x_0 \tag{10}$$

Within the region considered where x, y and z are positive, we easily deduce that $y_\infty = \lim_{t\to\infty} y(t) = 0$, which implies that the identity (3) holds.

The properties (ii) and (iii) of the theorem will be deduced by a direct and an approximation argument. We may safely omit the details.

We will utilize its outcomes to estimate the risk of epidemics.

B. Doubly Stochastic Poisson Process

As a time-dependent process for the trigger variable of epidemics, we here employ the so-called doubly stochastic Poisson process, or a Cox process (see [4]), which we recall briefly here.

The reason why we prefer a doubly-Poisson process to a usual homogeneous Poisson process is that the latter has deterministic intensity and hence it is rather inappropriate for modelling the resulting claims for epidemics outbreaks. On the other hand, A doubly stochastic Poisson process or a Cox process are known to provide flexibility in modelling, since the intensity process is allowed to be stochastic. As a general reference of stochastic processes including above processes, we refer to, for instance, a comprehensive book of Rolski, Schmidl, Schmidt, and Teugels [5].

Now we turn our attention to our process. Let $\Lambda = \{\lambda_t\}_{t\geq 0}$ be an intensity process; namely, a nonnegative, measurable, and locally integrable stochastic process. A counting process $\{N(t;\Lambda)\}_{t\geq 0}$ is called a Cox process or a doubly stochastic Poisson process with intensity Λ if for each sequence $\{k_i\}_{i=1,2,3,\cdots,n}$ of nonnegative integers, and for $0 < t_1 \leq s_1 \leq t_2 \leq s_2 \leq \cdots \leq t_{n-1} \leq s_{n-1} \leq t_n \leq s_n$, there holds

$$P\left(\bigcap_{i=1}^{n}\{N(s_i;\Lambda) - N(t_i;\Lambda) = k_i\}\right) = \prod_{i=1}^{n} E\left[\frac{1}{k_i!}\left(\int_{t_i}^{s_i}\lambda_u du\right)^{k_i} \exp\left(-\int_{t_i}^{s_i}\lambda_u du\right)\right]. \tag{11}$$

One typical example, which is favorably used to measure the effect of catastrophic events, and therefore suitable to epidemic bursts also, is the shot noise process. See for examples, Cox and Isham [6], Dassios and Jang [7], Kluppelberg and Mikosch [8]. Suppose $0 < s_1 < s_2 < \cdots$ are the points of a Poisson process with the intensity ρ and $\{y_i\}_{i=1,2,3,\cdots}$ are independently and identically distributed nonnegative random variables with $E[y_i] < \infty$, then

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{i:\, s_i \leq t} y_i e^{-\delta(t-s_i)}, \tag{12}$$

where λ_0 is the initial value and δ denote the rate of exponential decay. In Dassios and Jang [7] it is generalized that parameters ρ, δ are allowed to be time-dependent. We clearly imagine that s_i corresponds to the time at which catastrophe i occurs and y_i supplies the jump size of the catastrophe i. However, we will not treat this shot noise process anymore in the sequel.

The next section is devoted to the introduction of a model for estimating the risk concerning epidemic outbreaks, which is performed by the use of a doubly Poisson process or a Cox process.

III. Model

Now we introduce a model for estimating the risk of epidemic outbreaks.

The idea is direct and simple. We just combine risk factors of epidemics, market fluctuations, and others mentioned above; therefore, our model may possess the following features.

1) The outbreak of epidemics is modelled to be governed by a doubly stochastic Poisson or a Cox process.

This is due to the similarity between epidemics bursts and the occurrence of catastrophic events.

2) Once the outbreak of epidemics begins, then the situation would be like one described by the threshold theorem.

Of course, this part can be modified so that other types of epidemic modelling may be employed.

Anyway, in addition to these factors, we have to consider some kind of market movements.

3) Since insurance companies are involved in real markets, the market risk should be included.

This factor may be designed by the use of a stochastic process $S(t)$:

$$S(t) = S_0 \exp\left[-\alpha N(t) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right], \quad (13)$$

where $S_0 > 0$, $\alpha > 0$, $\mu > 0$ and the volatility σ are given positive constants. The process $S(t)$ is accompanied by the bond process $B(t) = e^{rt}$ with constant interest rate r. This is just a well-known Black-Scholes-Merton model. $N(t) := N(t; \Lambda)$ means a doubly stochastic Poisson process and $W(t)$ denotes the standard one-dimensional Brownian motion, which is independent of $S(t)$.

4) Insurance companies should take a specific policy for the insured.

Several such policies are known to be used. Here we appeal to the so-called stop-loss policy of the form

$$\max\{L - K, 0\}, \quad (14)$$

where L denotes the loss, and K is a retention level. Other policies are also possible.

To summarize above considerations, we are lead to the following model for the risk $R(t)$ of an insurer: $R(t)$ is the discounted value of the payoff function at the maturity T.

$$1_{\{N(T)>\nu\}} \cdot 2(N(T) - \nu) \cdot \max\{S(T) - K, 0\}, \quad (15)$$

where $\nu > 0$ denote the threshold value, and 1_Astands for the indicator function:

$$1_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \in A^C \end{cases}. \quad (16)$$

Here the factor $2(N(t) - \nu)$ comes from the threshold theorem (iii), and the factor $\max\{S(t) - K, 0\}$ may be interpreted that an insurer holds the strategy of the so-called stop-loss premium with a retention level K.

In the next section, we compute the discounted value of (1) under the risk neutral probability measure.

IV. Estimation Formula

A. Key Lemma

Here we analyze our model and derive a pricing formula for the risk of epidemic bursts.

In order to proceed, we need the next lemma, which is taken from [9]. See also [10].

Lemma. Let $\{N(t) = N(t; \Lambda)\}_{t\geq 0}$ be a doubly stochastic Poisson process with intensity $\Lambda = \{\lambda_t\}_{t\geq 0}$.

Then the process gives a martingale, where $k = 1 - e^{-\alpha}$ and $M_{\Lambda_t}(k)$ denotes the moment generating function of the aggregated process $\Lambda_t := \int_0^t \lambda_s ds$; namely

$$\left\{\exp\{-\alpha N(t) + \log(M_{\Lambda_t}(k))\}\right\}_{t\geq 0} \quad (17)$$

$$M_{\Lambda_t}(k) = E[\exp(k\Lambda_t)]. \quad (18)$$

Sketch of proof. Let $F_t^\Lambda := \sigma\{\lambda_s \mid 0 \leq s \leq t\}$. We wish to show that for $0 \leq s \leq t$

$$E\left[\exp\left\{-\alpha N(t) + \log\left(M_{\Lambda_t}(k)\right)\right\} \mid F_s^\Lambda\right] = \exp\{-\alpha N(s) + \log(M_{\Lambda_s}(k))\}. \quad (19)$$

To ensure this, we firstly learn that

$$E\left[e^{-\alpha N(t)}\right] = E\left[\sum_{l=0}^{\infty} e^{-\alpha l} P(N(t) = l)\right]$$
$$= E\left[\sum_{l=0}^{\infty} e^{-\alpha l} \frac{\Lambda_t^l}{l!} e^{-\Lambda_t}\right] = E[\exp(\Lambda_t(e^{-\alpha} - 1))]. \quad (20)$$

We then compute

$$\begin{aligned}
&E\left[\exp\left\{-\alpha N(t) + \log\left(M_{\Lambda_t}(k)\right)\right\} \mid F_s^\Lambda\right] \\
&= E\left[\exp\left\{-\alpha\big(N(t) - N(s)\big) + \log\left(M_{(\Lambda_t - \Lambda_s)}(k)\right)\right\} \mid F_s^\Lambda\right] \\
&\quad \cdot \exp\left\{-\alpha N(s) + \log\left(M_{\Lambda_s}(k)\right)\right\} \\
&= E\left[\exp\{-\alpha\big(N(t) - N(s)\big)\} \mid F_s^\Lambda\right] M_{(\Lambda_t - \Lambda_s)}(k) \\
&\quad \cdot \exp\left\{-\alpha N(s) + \log\left(M_{\Lambda_s}(k)\right)\right\} \quad (21) \\
&= M_{(\Lambda_t - \Lambda_s)}(-k) M_{(\Lambda_t - \Lambda_s)}(k) \\
&\quad \cdot \exp\left\{-\alpha N(s) + \log\left(M_{\Lambda_s}(k)\right)\right\} \\
&= \exp\{-\alpha N(s) + \log(M_{\Lambda_s}(k))\}.
\end{aligned}$$

This is what we wanted to know and the proof is completed.

B. Pricing Formula

Now we turn our attention to derive the risk pricing formula for an insurer.

We observe that the expected discounted value of (14) should be calculated under the risk-neutral measure Q. Precisely stated, our target is to evaluate

$$R(t) := E^Q\left[e^{-r(T-t)} 1_{\{N(T)>\nu\}} \cdot 2(N(T) - \nu) \cdot \max\{S(T) - K, 0\}\right]. \quad (22)$$

Now we learn that Q-mesaure is defined by

$$W^Q(t) := W^P(t) + \frac{(\mu - r)t + \log\left(M_{\Lambda_t}(k)\right)}{\sigma}, \quad (23)$$

where the notation $W^P(t) := W(t)$ is just used for the sake of avoiding a confusion. The Radon-Nikodym derivative process is

$$\frac{dQ}{dP} = \exp\left\{-\frac{1}{2}\int_0^t \left(\frac{\mu - r + \gamma_k(s)}{\sigma}\right)^2 ds - \int_0^t \frac{\mu - r + \gamma_k(s)}{\sigma} dW^P(s)\right\}. \quad (24)$$

Here $\gamma_k(t) := E[\lambda_t \exp(k\Lambda_t)]/M_{\Lambda_t}(k)$.

Consequently. in view of the key lemma, we see that the process $e^{-rt}S(t)$ defined by gives a Q-martingale. Thus we are able to compute the discounted value of (14) under Q-measure in a similar way as in [6].

$$e^{-rt}S(t) = S_0 \exp\left[-\alpha N(t) + \log\left(M_{\Lambda_t}(k)\right)\right] \cdot \exp\left[\sigma W^Q(t) - \frac{1}{2}\sigma^2 t\right] \tag{25}$$

To summarize, we have established our main theorem as follows.

Theorem. The risk $R(t)$ for an insurer, whose payoff function at the maturity T is (14) can be expressed as

$$R(t) = \sum_{l=\nu+1}^{\infty} 2(l-\nu)\left(S(t)e^{-\alpha l + \log\left(M_{(\Lambda_T-\Lambda_t)}(k)\right)}\Phi\left(d_l + \sigma\sqrt{T-t}\right) - Ke^{-r(T-t)}\Phi(d_l)\right) \cdot E\left[\frac{(\Lambda_T-\Lambda_t)^l}{l!}e^{-(\Lambda_t-\Lambda_t)}\right], \tag{26}$$

where $k = 1 - e^{-\alpha}$, *and*

$$d_l = \frac{\log\left(\frac{S(t)}{K}\right) + r(T-t) - \alpha l + \log\left(M_{(\Lambda_T-\Lambda_t)}(k)\right)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}\,. \tag{27}$$

Here $\Phi(d)$ denotes the cumulative distribution function for the standardized normal distribution

$$\Phi(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{x^2}{2}}dx\,. \tag{28}$$

It is easy to see that the quantities (25), (26) are modifications of the famous Black-Scholes option pricing formula for the vanilla European call options.

The proof proceeds along the same line as in [9] and involves a calculation of relevant quantities; we may safely omit the details.

C. Hedging Parameters

For the real world applications, various hedging parameters are important, whose calculation is provided straightforwardly. We here just present the results without a proof, which may be directly performed through rather standard fashion.

Proposition. Let the risk $R(t)$ for an insurer, whose payoff function at the maturity T is (14) be given by (25). Then the Delta, Gamma, Rho, and Vega of $R(t)$ are computed as the next expressions, respectively.

$$\Delta(t) = \frac{\partial R(t)}{\partial S} = \sum_{l=\nu+1}^{\infty} 2(l-\nu)e^{-\alpha l + \log\left(M_{(\Lambda_T-\Lambda_t)}(k)\right)}\Phi\left(d_l + \sigma\sqrt{T-t}\right) \cdot E\left[\frac{(\Lambda_T-\Lambda_t)^l}{l!}e^{-(\Lambda_t-\Lambda_t)}\right], \tag{29}$$

$$\Gamma(t) = \frac{\partial^2 R(t)}{\partial S^2} = \sum_{l=\nu+1}^{\infty} 2(l-\nu)\frac{e^{-\alpha l + \log\left(M_{(\Lambda_T-\Lambda_t)}(k)\right)}}{S\sigma\sqrt{T-t}}\varphi\left(d_l + \sigma\sqrt{T-t}\right) \cdot E\left[\frac{(\Lambda_T-\Lambda_t)^l}{l!}e^{-(\Lambda_t-\Lambda_t)}\right], \tag{30}$$

$$\rho(t) = \frac{\partial R(t)}{\partial r} = \sum_{l=\nu+1}^{\infty} 2(l-\nu)K(T-t)\Phi(d_l) \cdot E\left[\frac{(\Lambda_T-\Lambda_t)^l}{l!}e^{-(\Lambda_t-\Lambda_t)}\right], \tag{31}$$

$$\nu(t) = \frac{\partial R(t)}{\partial \sigma} = \sum_{l=\nu+1}^{\infty} 2(l-\nu)S(T-t)\varphi\left(d_l + \sigma\sqrt{T-t}\right) \cdot E\left[\frac{(\Lambda_T-\Lambda_t)^l}{l!}e^{-(\Lambda_t-\Lambda_t)}\right], \tag{32}$$

where d_l is defined by (26) and $\varphi = \Phi'$.

V. Conclusion

In this paper, we have developed a simple stochastic model for an insurer which takes part in mitigating the effect of unexpected sudden epidemic outbreaks. The onset of pandemic is modelled by a double stochastic Poisson process or a Cox process. After the trigger, the ongoing of epidemics follows the story described by the Kermack-Mckendrick threshold theorem.

Additionally, we assume that the insurer is exposed to the market risk, which is governed by the mixture of above jump process and the usual Black-Scholes-Merton model.

We then estimate the risk of an insurer subject to above mentioned risk factors. The method is based on a similar argument as employed in the case of catastrophic options. The closed form solution is obtained. It is to be noted that some empirical investigations should be undertaken, which will be our future project of researches.

Despite of its importance, the study on insurance attitude against certain epidemics is not so popular in the literature; especially, on the basis of mathematical modelling. Observe for instance Tamura and Sawada [11] as an exceptional example of different viewpoint. We believe that the financial aspect of these unhappy events should be pursued further. We hope that our first step attempt will open a fairly wide area of both academic and practical researches.

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