A New General Approach to Vector Valued Stochastic Integration

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Abstract—**We use an extended theory of integral that generalizes the integration of vector valued functions with respect to non-negative, monotonic, countably subadditive set functions, in order to introduce a new approach to stochastic integral. With such an approach, we will explore the possible extension of the theory of stochastic integration to the more general setting of integrable processes taking values in normed vector spaces. We show that our approach makes applications possible to stochastic processes that are not necessarily square integrable, nor even measurable. Such an extension generally consolidates the typical and classical results obtained for the standard scalar case.**

Index Terms—**Vector integration, banach spaces, stochastic processes, martingales, conditional expectation, tensor product.**

I. Introduction

The theory of stochastic integration has become an intensely studied topic in recent years owing to its extraordinarily successful application to the growing area of financial mathematics, stochastic differential equations, probability, and more. The classical theory of stochastic integration for real valued processes essentially reduces to integration with respect to a square integrable martingale. This is usually done by defining the stochastic integral for simple processes and then extending it to a more general class of processes by approximation assisted by some Hilbert-space functional analysis. Some research results in the last decade have shown obvious need for the extension of stochastic integration to vector valued setting (see for example Maas [1], Noreddine and Nourdin [2]). It becomes obvious that a more rigorous approach to a more general definition of the stochastic integral need to be developed. Many attempts have already been made in this direction (see for example Kussmaul [3], Pellaumail [4]). However, a number of subtleties still arise when the integral is intended to satisfy additional more abstract properties. In his book, Dinculeanu [5] introduces a measure theoretic approach to stochastic integration. He first develops a general integration theory with respect to vector measures with finite semivariation. To a Banach-valued process X, he associates a measure I_X on the ring generated by the predictable rectangular sets. If I_X can be extended to a σ-additive measure with finite semivariation on the σ-algebra of predictable sets, then he the process X is said to be summable. Roughly speaking, he defines the stochastic integral of a process H with respect to X to be the process $\int_{[0,t]} HdI_X$ of integrals with respect to I_X. Such an approach is relatively satisfactory but also highly technical and therefore requires a mastery of the full theory of vector measures and the theory of integration with respect to vector measures.

It is our aim in this note to introduce a new approach to stochastic integration that is simpler yet more general and more efficient than the classical theory. We first define a stochastic process as any well-ordered family of vector-valued extended integrable functions in the sense of integration theory first introduced in Robdera [6], and further developed in Robdera [7]. The stochastic integral of a process with respect to another process is then simply defined as the net-limit of some 'stochastic Riemann sums' defined for one of the processes with respect to the other. The theory of stochastic integral introduced in this paper extends the classical case with the advantage of not requiring any Hilbert-space functional analysis making applications possible to the cases of more abstract vector-valued integrable processes. The originality of our approach also stems from the fact that it frees stochastic integration from the heavy machinery of the vector measure theory. Such a new approach naturally leads to a further opening to the field of research in stochastic processes.

The reader is supposed to be familiar with the classical theory of integral as well as the general theory of stochastic processes for the scalar case presented in any elementary textbook on stochastic processes, for example Dellacherie [8].

II. Extended Integration Theory

For the sake of clarity and completeness, we summarize in this section, the main points of the basic concepts in the extension of the theory of integration introduced and developed in Robdera [7].

Let V be a normed vector space and let a measure space (Ω, Σ, μ), be given. We denote by μ^* the outer-measure obtained by the Carathéodory extension of the measure \mu to the whole of the power set 2^Ω. The restriction of $\mu^*$to any σ-algebra Σ' containing the σ-algebra Σ will also be denoted by μ^*. Such an extension satisfies

- $\mu^*(\emptyset) = 0$;
- $\mu^*(A) \leq \mu^*(B)$ whenever $A \subset B$ in Σ' (monotonicity);
- $\mu^*(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n)$ for every sequence $n \mapsto A_n$ in Σ' such that $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma'$ (countable subadditivity).

We shall call any arbitrary set function defined on a σ-ring satisfying the above three conditions a size function. Thus a measure and its Carathéodory extension are examples of size

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functions.

The following proposition is easily verified.

Proposition 1. Let $\Sigma_1 \subset \Sigma_2$ be two σ-algebras. Let μ be a measure. The Carathéodory extension defined for every subset $A \in \Sigma_2$ by is a size function defined on Σ_2 and satisfies

$$\mu^*(A) = \inf\{\textstyle\sum_{n=1}^{\infty} \mu(I_n): A \subset \bigcup_{n\in\mathbb{N}} I_n,\ I_n \in \Sigma_1\} \quad (1)$$

$$\mu^*(E) = \mu(E)\ for\ every\ E \in \Sigma.$$

We now recall the main ideas of the construction of the integral with respect to μ^*.

By a Σ-subpartition of a subset A of Ω, we mean any finite collection satisfying $I_i \cap I_j$ whenever $i \neq j$. We shall denote by $\bigcup P$ the subset of A obtained by taking the union of all elements of P.

A Σ-subpartition $P = \{I_i: I_i \subset A, i = 1,2,\dots,n\}$ is said to be tagged if a point $t_i \in I_i$ is chosen for each $i \in \{1,2,\dots,n\}$.

$$P = \{I_i: I_i \subset A, i = 1,2,\dots,n\}$$

We shall write if we wish to specify the tagging points. We denote by $\Pi(\mathrm{A},\Sigma)$ the collection of all tagged Σ-subpartitions of the set A

$$P = \{(I_i, t_i), i = 1,2,\dots,n\}$$

The mesh or the norm of the subpartition $P \in \Pi(\mathrm{A},\Sigma)$ with respect to μ^* is defined to be

$$\|P\| = \max\{\mu^*(I_i): I_i \in P\}. \quad (2)$$

If $P, Q \in \Pi(\mathrm{A},\Sigma)$, we say that Q is a refinement of P and we write $Q \succ P$ if $\|Q\| \leq \|P\|$ and $\bigcup P \subset \bigcup Q$.

It is readily seen that such a relation does not depend on the tagging points. It is also easy to see that the relation $\succ$ is transitive on $\Pi(\mathrm{A},\Sigma)$.

Given $P, Q \in \Pi(\mathrm{A},\Sigma)$, we shall denote the subpartition $P \vee Q \coloneqq \{I \setminus J, I \cap J, J \setminus I: I \in P, J \in Q\}$.

Clearly, $P \vee Q \in \Pi(\mathrm{A},\Sigma)$, $P \vee Q \succ P$, and $P \vee Q \succ Q$. That is, the binary relation $\succ$ has the upper bound property on $\Pi(\mathrm{A},\Sigma)$.

We then infer that the set $\Pi(\mathrm{A},\Sigma)$ is directed (in the sense defined in McShane [4]) by the binary relation $\succ$.

Given a function $f: \Omega \longrightarrow V$, and a tagged Σ-subpartition, we define the Σ-Riemann sum (with respect to μ^*) of f at P to be the vector

$$\mu^*{}_f(P) = \textstyle\sum_{i=1}^{n} \mu^*(I_i) f(t_i). \quad (3)$$

Thus the function $P \longmapsto \mu^*{}_f(P)$ is a net defined on the directed set $(\Pi(\mathrm{A},\Sigma), \succ)$ taking values in the normed vector space V.

For convenience, we are going to denote the net-limit by

$$\textstyle\int_A f d\mu^* \coloneqq \lim_{(\Pi(\mathrm{A},\Sigma),\succ)} \mu^*{}_f \quad (4)$$

whether or not such a limit exists. For details on net-limit werefer the reader to McShane [9].

The notion of integrability with respect to the size function is defined as follows.

Definition 2. We say that a function $f: \Omega \longrightarrow V$ is Σ-integrable over a subset A of Ω, if the limit $\int_A f d\mu^*$ represents a vector in V.

The vector $\int \int_A f d\mu^*$ is then called the Σ-integral of over the set A.

In other words, $f: \Omega \longrightarrow V$ is Σ-integrable over the set with Σ-integral $f: \Omega \longrightarrow V$ if for every $\epsilon > 0$, there exists a Σ-subpartition P_0 of the set A such that for every $P \succ P_0$ in $\Pi(\mathrm{A},\Sigma)$, we have

$$\left\| \textstyle\int_A f d\mu^* - \mu^*{}_f(P) \right\|_V < \epsilon. \quad (5)$$

For every measure, we shall simply denote by $\mathcal{I}(A,\Sigma,V)$ the set of all (Σ,μ^*)-integrable functions over the set A. If V is the scalar field, we simply write $\mathcal{I}(A,)$ instead of where $\mathcal{I}(A,\Sigma,\mathbb{K})$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

One of the advantages of the above definition of the integral is the fact that no notion of measurability needs to be postulated. This approach allows us to free integration theory from the measure theory.

It is easily verified that if Σ_1 and Σ_2 are two σ-algebras in 2^Ω such that $\Sigma_1 \subset \Sigma_2$, then clearly

$$\Pi(A,\Sigma_1) \subset \Pi(A,\Sigma_2) \quad (6)$$

Hence we have the following proposition stating the relationship between Σ_1-integrability and Σ_2-integrability.

Proposition 3. Let $\Sigma_1 \subset \Sigma_2$, be two σ-algebras in 2^Ω. Then for every $A \in 2^\Omega$, for $f \in \mathcal{I}\left(A,\Sigma,\mu^*{}_{|\Sigma_1} V\right)$

$$\mathcal{I}\left(A,\Sigma,\mu^*{}_{|\Sigma_1},V\right) \subset \mathcal{I}\left(A,\Sigma,\mu^*{}_{|\Sigma_2},V\right) \quad (7)$$

And

$$\textstyle\int_A f d\mu^*{}_{|\Sigma_1} = \int_A f d\mu^*{}_{|\Sigma_2} \quad (8)$$

In view of Proposition 1 and Proposition 3, we can always consider integrability with respect to the outer-measure μ^* on the whole of 2^Ω and we shall write $\int_A f d\mu^*$ instead of $\int_A f d\mu^*{}_{|\Sigma_1}$.

It should be clear from the above proposition that Lebesgue-Bochner integrable functions with respect to a measure defined on a σ-algebra are integrable with respect to the Carathéodory extension μ^* of μ on any σ-algebra Σ_2 containing Σ_1.

As shown in Robdera [10], if V is a Banach space, the space $\mathcal{I}(A,\Sigma,V)$ can be given a structure of complete seminormed space with respect to the seminorm defined for every $f \in \mathcal{I}(A,\Sigma,V)$ by

$$\|f\|_{\Pi,V} \coloneqq \sup\left\{\left\|\mu^*{}_f(P)\right\|_V : P \in \Pi(\Omega,\Sigma)\right\}.$$

It follows that the integral does not distinguish between functions which differ only on set of size zero. More precisely,

$$\int_A f d\mu^* = \int_A g d\mu^* \tag{9}$$

whenever $\mu^*\{x \in A: f(x) \neq g(x)\} = 0$.

We say that a function f is μ^*-essentially equal on A to another function g, and we write $f \sim g$, if

$$\mu^*\{x \in A: f(x) \neq g(x)\} = 0. \tag{10}$$

It is readily seen that the relation is an equivalence relation on $\mathcal{I}(A, \Sigma, V)$. We shall denote by $I(A, \Sigma, V)$ the quotient space $\mathcal{I}(A, \Sigma, V)/\sim$.

It goes without saying that when the μ^*-equivalent functions taking values in a Banach space are identified, the restriction of the seminorm $\|\cdot\|_{\Pi,V}$ defines a norm on $I(A, \Sigma, V)$, and gives the space $I(A, \Sigma, V)$ the structure of a Banach space.

In what follows, by integrable functions we shall always mean μ^*-integrable functions, i.e. functions in the vector space $\mathcal{I}(A, \Sigma, V)$.

For every $1 \leq p$, we shall call I^p**-spaces** and we denote by $I^p(A, \Sigma, V)$ the vector space of all (not necessarily measurable) functions such that the scalar function $\omega \longmapsto (\|f(\omega)\|_V)^p$ is in $I(A, \Sigma)$. Such a space will be endowed with the norm defined by

$$\|f\|_p = \left(\int_A (\|f(\cdot)\|_V)^p d\mu^*\right)^{1/p} \tag{11}$$

It is easily checked that the Minkowski's inequality and the Hölder's inequality hold for the I^p-spaces. The classical proof of the fact that the space of Lebesgue p-integrable functions $L^p(A, V)$ can be paralleled to show that $I^p(A, \Sigma, V)$ is complete with respect to the seminorm $\|\cdot\|_p$

The classical Riemann approach to integral lacks the all important convergence properties of the Lebesgue integral. However, it is shown in Robdera [7] that the Lebesgue convergence theorems, namely the Fatou's Lemma, the monotone and the dominated convergence theorems all have their counterparts in the space of extended integrable functions $I(A, \Sigma, V)$.

III. Conditional Expectation, Stochastic Process, Martingales

In this section, we extend some of the basic notions in the general theory of stochastic processes.

Again, V will denote a normed vector space. We have seen in Proposition 3 that if $\Sigma_1 \subset \Sigma_2$, are two σ-algebras of elements in 2^Ω, then for every $A \in 2^\Omega$, and if $f \in I(A, \Sigma_1, V)$ then

$$I(A, \Sigma_1, V) \subset I(A, \Sigma_2, V) \tag{12}$$

$$\int_A f d\mu^*_{|\Sigma_1} = \int_A f d\mu^*_{|\Sigma_2} \tag{13}$$

Conversely, assume that $f \in \mathcal{I}(A, \Sigma_2, V)$. Then the map is an additive set function. According to the extended version of the Lebesgue-Nikodým Theorem proved in Robdera [10], there exists a μ^*-almost unique function $g \in \mathcal{I}(A, \Sigma_1, V)$ such that

$$\begin{array}{rccc} \eta: & \Sigma_1 & \longrightarrow & V \\ & A & \longmapsto & \int_A f d\mu^* \end{array} \tag{14}$$

$$\eta(A) = \int_A g d\mu^*. \tag{15}$$

Such a function is called the **conditional expectation** of f given Σ_1 and is denoted by $E(f|\Sigma_1)$.

Remark. Note that as opposed to the classical definition of conditional expectation, here the measurability of the function $E(f|\Sigma_1)$ is not required. However, it is easily checked that the basic properties of the classical conditional expectation are preserved, namely:-

If $\in I(A, \Sigma_2, V)$, then $E(f|\Sigma_1) \sim f$.

If $\Sigma_1 \subset \Sigma_2 \subset \Sigma$ are σ-algebras, $f \in I(A, \Sigma, V)$, then

$$E(E(f|\Sigma_1)|\Sigma_2) \sim E(f|\Sigma_1). \tag{16}$$

If $\Sigma_1 \subset \Sigma_2$ are σ-algebras, $f, g \in \mathcal{I}(A, \Sigma_2, V)$ and α, β scalars, then

$$E(\alpha f + \beta g|\Sigma_1) \sim \alpha E(f|\Sigma_1) + \beta E(f|\Sigma_1). \tag{17}$$

We define a stochastic process as an ordered family of integrable functions:

Definition 4. Let V be a normed space. *A* V-valued stochastic process is any function $X: T \longrightarrow I(\Omega, \Sigma, V)$ where T is an ordered set.

Again in this definition, no notion of measurability is required. However, we shall say that the stochastic process is Σ-measurable if for every $t \in T$, the function $t \longmapsto X(t)$ is Σ-measurable.

A stochastic process is said to be discrete if T is the set of all natural numbers, and continuous if T is uncountable. The usual and typical examples are given by $T = \mathbb{N}$, $T = [0, \infty)$, and $T = [a, b] \subset \mathbb{R}$.

A stochastic process $\Theta: T \longrightarrow I(\Omega, \Sigma, V)$ is said to be a nil-process over a subset S of T if for each $s \in S$, $\Theta(s) \sim 0$ in $I(\Omega, \Sigma, V)$.

A stochastic process $X: T \longrightarrow I(\Omega, \Sigma, V)$ is said to be p-integrable over a subset S of T if for each $\in S$, $X(s) \in I^p(\Omega, \Sigma, V)$. It is worth mentioning that most of the processes in the classical scalar case are 2-integrable. For example, the Brownian process is 2-integrable.

An important class of stochastic processes is given by the so-called martingales. First recall that a filtration is a family $\{\Sigma_t: t \in T\}$ of σ-algebras in Σ satisfying $\Sigma_s \subset \Sigma_t$ whenever $s < t$. A stochastic process is said to be *adapted* to a filtration $\{\Sigma_t: t \in T\}$ if for each $t \in T$, $X(t) \in I(\Omega, \Sigma_t, V)$.

We extend the definition of a martingale as follows.

Definition 5. A martingale is a stochastic process $X: T \longrightarrow I(\Omega, \Sigma, V)$ adapted to some filtration $\{\Sigma_t: t \in T\}$ that satisfies *for all* $s \leq t$.

$$E(X(t)|\Sigma_s) = X(s) \tag{18}$$

Here again, as opposed to the classical case, neither the absolute integrability nor the measurability of the functions involved is required.

IV. Stochastic Integration

In what follows, the projective tensor product of two normed spaces V and W shall be denoted by $V\widehat{\otimes}W$. For details on tensor products of normed spaces, the reader is referred to Ryan [11]. We fix a measure space (Ω, Σ, μ), and a well-ordered index set T.

Given $a < b$ in T, the set of all $t \in T$ such that $a \leq t \leq b$ will be denoted by $[a, b]$ and will be called interval.

Definition 6. *Let Vand Wbe normed vector spaces. Let* $Y: T \longrightarrow I(\Omega, \Sigma, V)$ *and* $X: T \longrightarrow I(\Omega, \Sigma, W)$ be two stochastic processes. Given a partition of the interval $[a, b]$, we define the stochastic Riemann sum of Y with respect to X at π to be the tensor.

$$\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

$$Y_X(\pi) = \sum_{i=1}^{n} Y(t_{i-1}) \otimes (X(t_i) - X(t_{i=1})) \quad (19)$$

It is easily verified that $Y_X(\pi) \in I(\Omega, \Sigma, V\widehat{\otimes}W)$ for every partition π of $[a, b]$.

Remark 7. For the particular case where $V = W$ is a multiplicative normed space, that is to say, - to every pair $(u, v) \in V \times V$, one can associate a vector $uv \in V$ in such a way that $\|uv\| \leq \|u\|\|v\|$ -, we simply use the tensor product defined by $(u, v) \longrightarrow u \otimes v = uv$. The stochastic Riemann sum of Y with respect to X is then given by

$$Y_X(\pi) = \sum_{i=1}^{n} Y(t_{i-1})(X(t_i) - X(t_{i=1})) \quad (20)$$

which is obviously an element of $i(\Omega, \Sigma, V\widehat{\otimes}V)$.

Such is the case, for example when $V = W = \mathbb{K}$, or more generally, when $V = W$has the structure of a Banach algebra.

We shall denote by $\pi([a, b])$ the set of all partitions of directed by the usual partition refinement $\succ$. It follows that $[a, b]$ the map is a net. We let $\int_a^b YdX := \lim_{\succ} Y_X(\pi)$ whether or not such a limit exists.

$$\begin{array}{rccc} Y_X: & \pi([a,b]) & \longrightarrow & I(\Omega, \Sigma, V\widehat{\otimes}W) \\ & \pi & \longmapsto & Y_X(\pi) \end{array} \quad (21)$$

We are now ready to give our definition of stochastic integral.

Definition 8. Let Vand Wbe normed spaces. Let $Y: T \longrightarrow I(\Omega, \Sigma, V)$ and $X: T \longrightarrow I(\Omega, \Sigma, W)$ be two stochastic processes. We say that Y is **stochastically integrable** with respect to X over the interval $[a, b]$ if the limit $\int_a^b YdX$ represents an element in $I(\Omega, \Sigma, V\widehat{\otimes}W)$

The limit $\int_a^b YdX$ is then called the stochastic integral of $Y$with respect to X over $[a, b]$.

In other words, the process Y is stochastically integrable with respect to the process X over the interval $[a, b]$, if there exists an element $\int_a^b YdX$ in $I(\Omega, \Sigma, V\widehat{\otimes}W)$ such that for every $\epsilon > 0$, there exists $\pi_0 \in \pi([a, b])$ such that if $\pi \succ \pi_0$ in $\pi([a, b])$, then

$$\left\| \int_a^b YdX - Y_X(\pi) \right\|_{\Pi, V\widehat{\otimes}W} < \epsilon \quad (22)$$

It follows immediately from the linearity of the net-limit that the stochastic integration is a linear operator; that is to say, if both the processes $Y, Z: T \longrightarrow I(\Omega, \Sigma, V)$ are both integrable over $[a, b] \in T$, with respect to the same process $X: T \longrightarrow I(\Omega, \Sigma, W)$ and if α and β are any pair of scalars, then

$$\int_a^b (\alpha Y + \beta Z)dX = \alpha \int_a^b YdX + \beta \int_a^b YdX. \quad (23)$$

It also follows from the stochastic Riemann sum that if $Y = 1_{[a,b]}x$ where $1_{[a,b]}$ is the indicator function of $[a, b]$, and x a constant vector in V, then

$$\int_a^b YdX = x \otimes [X(b) - X(a)]. \quad (24)$$

We notice that if $Y: T \longrightarrow I(\Omega, \Sigma, V)$ is stochastically integrable over an interval $[a, b] \subset T$ with respect to $X: T \longrightarrow I(\Omega, \Sigma, W)$ then it is stochastically integrable over any subinterval [c,d] subset $[a, b]$.

If $Y: T \longrightarrow I(\Omega, \Sigma, V)$ is stochastically integrable with respect to $X: T \longrightarrow I(\Omega, \Sigma, W)$ over $[a, b]$ then defines another stochastic process that we shall call the indefinite stochastic integral of Y with respect to X over the interval $[a, b]$.

$$\begin{array}{rccc} Z: & [a,b] & \longrightarrow & I(\Omega, \Sigma, V\widehat{\otimes}W) \\ & u & \longmapsto & \int_a^u YdX \end{array} \quad (25)$$

As in the classical theory, the case of martingales presents a particular feature.

Theorem 9. The indefinite stochastic integral of a martingale with respect to a martingale adapted to the same filtration is a martingale adapted to the same filtration.

Proof. Let $u \longmapsto Z(u) = \int_a^u YdX$ be the indefinite integral of a martingale $Y: T \longrightarrow I(\Omega, \Sigma, V)$ with respect to a given martingale $X: T \longrightarrow I(\Omega, \Sigma, W)$ adapted to the same filtration as Y. We notice that for $s < t$,

$$\begin{aligned} &E\big((Z(t) - Z(s))|\Sigma_s\big) \\ &= E\left(\left(\int_{a_0}^t YdX - \int_{a_0}^s YdX\right)|\Sigma_s\right) \\ &= E\left(\left(\int_s^t YdX\right)|\Sigma_s\right) \end{aligned} \quad (26)$$

Fix $\epsilon > 0$. There exists $\pi_0 \in \pi([s, t])$ such that if $\pi \succ \pi_0$ in $\pi([s, t])$,

$$\left\| \int_s^t YdX - Y_X(\pi) \right\|_{\Pi, V\widehat{\otimes}W} < \epsilon. \quad (27)$$

We note that whenever $\pi \succ \pi_0$ in $\pi([s, t])$, then $Y(t_{i=1}) \in I(\Omega, \Sigma_s, W)$ and thus

$$E(Y(t_{i=1})|\Sigma_s) = Y(t_{i=1}). \quad (28)$$

It follows that and hence

$$\begin{aligned} &E[\textstyle\sum_{i=1}^{n} Y(t_{i=1}) \otimes (X(t_i) - X(t_{i=1}))|\Sigma_s] \\ &= \textstyle\sum_{i=1}^{n} E[Y(t_{i=1}) \otimes (X(t_i) - X(t_{i=1}))|\Sigma_s] \\ &= \textstyle\sum_{i=1}^{n} Y(t_{i=1}) \otimes E[(X(t_i) - X(t_{i=1}))|\Sigma_s] \\ &= 0 \end{aligned} \quad (29)$$

$$\begin{aligned} &\left\|E\left(\left(\int_s^t YdX\right)|\Sigma_s\right)\right\| \\ &= \left\|E\left(\left(\int_s^t YdX - Y_X(\pi)\right)|\Sigma_s\right)\right\| \\ &\leq E\left[\left\|\int_s^t YdX - Y_X(\pi)\right\| |\Sigma_s\right] \\ &< \epsilon. \end{aligned} \tag{30}$$

Since $\epsilon > 0$ is arbitrary, $E\left(\left(\int_s^t YdX\right)|\Sigma_s\right) = 0$. It follows from (26) that $E\big((Z(t) - Z(s))|\Sigma_s\big) = 0$ or equivalently $E(Z(t)|\Sigma_s)$=Z(s). This completes the proof.

V. A Convergence Theorem

In this section, we assume that the normed vector spaces V and $W$are both Banach spaces (real or complex). We say that a process $X: T \longrightarrow I(\Omega, \Sigma, W)$ is of bounded variation if is finite, where the supremum is taken over all partitions

$$var(X) = \sup\{\textstyle\sum_{i=1}^n \|X(t_i) - X(t_{i-1})\|_{\Pi,W} \tag{31}$$

$$\pi = \{a = t_0 < t_1 < \cdots < t_n = b\} \in \pi([a,b]).$$

We denote by $\mathcal{S}_X([a,b], I(\Omega,\Sigma,V))$ the space of all processes $Y: T \longrightarrow I(\Omega,\Sigma,V)$ stochastically integrable with respect to the same process X over an interval $[a,b]$. It follows immediately from the properties of limit that the space $\mathcal{S}_X([a,b], I(\Omega,\Sigma,V))$ has the structure of a vector space.

We define

$$\|Y\|_{\mathcal{S}_X} = \sup\left\{\left\|\int_a^t YdX\right\|_{\Pi, V\hat{\otimes}W} : t \in [a,b]\right\} \tag{32}$$

It is easily checked that the map $Y \longmapsto \|Y\|_{\mathcal{S}_X}$ defines a seminorm on $\mathcal{S}_X([a,b], I(\Omega,\Sigma,V))$.

Our next result is a convergence theorem.

Theorem 10. Let V and W be two Banach spaces. Let $X: T \longrightarrow I(\Omega,\Sigma,W)$ be a process of bounded variation. Let $n \longmapsto Y_n$ be a sequence of elements of the linear space $\mathcal{S}_X([a,b], I(\Omega,\Sigma,V))$. Assume that for every$t \in [a,b]$, the sequence $n \longmapsto Y_n$ is Cauchy in $I(\Omega,\Sigma,V)$;

The sequence $n \longmapsto Y_n$ is Cauchy in $\mathcal{S}_X([a,b], I(\Omega,\Sigma,V))$. Then the sequence $n \longmapsto Y_n$ converges in

$$\mathcal{S}_X([a,b], I(\Omega,\Sigma,V)).$$

Proof. Since V is a Banach space, the space $I(\Omega,\Sigma,V)$ is complete. We then can define a process by

$$\begin{aligned} Y:\ [\mathrm{a,b}] &\longrightarrow I(\Omega,\Sigma,V) \\ t &\longmapsto \lim_{n\to\infty} Y_n(t) \end{aligned} \tag{33}$$

We are going to show that the sequence $n \longmapsto Y_n$ converges to the process $Y \in \mathcal{S}_X([a,b], I(\Omega,\Sigma,V))$.

We first claim that the sequence $n \longmapsto \int_a^t Y_n dX$ is Cauchy in $I\big(\Omega,\Sigma, V\hat{\otimes}V\big)$.

Fix $\epsilon > 0$. By the Cauchy condition on the sequence $n \longmapsto Y_n$, there exists N_ϵ such that for $n, m > N_\epsilon$, we have $\|Y_n - Y_m\|_{\mathcal{S}_X} < \epsilon$.

Since for every $t \in [a,b]$, both the processes Y_n and Y_m belong to $\mathcal{S}_X([a,b], I(\Omega,\Sigma,V))$, there exist two partitions π_n and π_m in $\pi([a,t])$, such that whenever $\pi_t \succ \pi_n$

$$\left\|Y_{n,X}(\pi_t) - \int_a^t Y_n dX\right\| < \epsilon \tag{34}$$

And whenever $\pi_t \succ \pi_m$

$$\left\|Y_{m,X}(\pi_t) - \int_a^t Y_m dX\right\| < \epsilon. \tag{35}$$

Note that $\pi_m \vee \pi_m \in \pi([a,b\})$. It follows that for $n, m > \epsilon$ in $\mathbb{N}$ and for every subpartition $\pi \succ \pi_m \vee \pi_m$ in $\pi([a,b\})$, we have

$$\begin{aligned} &\left\|\int_a^t Y_n dX - \int_a^t Y_m dX\right\| \\ &\leq \left\|Y_{n,X}(\pi_t) - \int_a^t Y_n dX\right\| + \left\|Y_{n,X}(\pi_t) - Y_{m,X}(\pi_t)\right\| \\ &+ \left\|Y_{m,X}(\pi_t) - \int_a^t Y_m dX\right\| \\ &< 2\epsilon + \|Y_n - Y_m\|_{\mathcal{S}_X} \end{aligned} \tag{36}$$

This proves our claim.

Since V and W are Banach spaces, the space $I\big(\Omega,\Sigma, V\hat{\otimes}V\big)$ is complete.

It follows that the Cauchy sequence $n \longmapsto \int_a^t Y_n dX$ converges to an element $Z(t) \in I\big(\Omega,\Sigma, V\hat{\otimes}V\big)$.

We can then define the process

$$\begin{aligned} Z:\ [\mathrm{a,b}] &\longrightarrow I\big(\Omega,\Sigma, V\hat{\otimes}W\big) \\ t &\longmapsto Z(t) \end{aligned} \tag{37}$$

On the other hand, by our hypothesis there exists $N_t > N_\epsilon$ such that for natural numbers m, n satisfying $m, n > N_t$,

$$\|Y_n(t) - Y_m(t)\|_{\Pi,V} < \frac{\epsilon}{var(X)} \tag{38}$$

Thus for each partition and for every pairs of natural numbers

$$\pi = \{a = t_0 < t_1 < \cdots < t_n = t\},$$

$$m, n > \max\{N_{t_i} : i = 1, \ldots, k\} := N_\pi,$$

We have

$$\begin{aligned} &\left\|Y_{n,X}(\pi_t) - Y_{m,X}(\pi_t)\right\| \\ &= \|\textstyle\sum_{i=1}^n [Y_n(t_{i-1}) - Y_n(t_{i-1})] \otimes [X(t_i) - X(t_{i=1})]\| \\ &\leq \textstyle\sum_{i=1}^n \|Y_n(t_{i-1}) - Y_n(t_{i-1})\| \otimes \|X(t_i) - X(t_{i=1})\| \\ &\leq \frac{\epsilon}{var(X)} var(X) = \epsilon \end{aligned} \tag{39}$$

If we let $m \longrightarrow \infty$ in the above inequality, we obtain

$$\left\|Y_{n,X}(\pi_t) - Y_{m,X}(\pi_t)\right\| < \epsilon \tag{40}$$

$$\begin{aligned} &\|Y_X(\pi_t) - Z(t)\| \\ &\leq \left\|Y_X(\pi_t) - Y_{n,X}(\pi_t)\right\| + \left\|Y_{n,X}(\pi_t) - \int_a^t Y_n dX\right\| \\ &+ \left\|\int_a^t Y_n dX - Z(t)\right\| < 3\epsilon \end{aligned} \tag{41}$$

Since $Z(t) = \lim_{n\to\infty} \int_a^t Y_n dX$, there exists $N > N_\pi$ such that for $n > N$. It follows that for $n > N$,

$$\left\|\int_a^t Y_n dX - Z(t)\right\| < \epsilon$$

Since $\epsilon > 0$ is arbitrary, this shows that $\int_a^t YdX = Z(t)$. It follows that $\in \mathcal{S}_X([a,b], I(\Omega,\Sigma,V\widehat{\otimes}W))$.

By (39), for $n, m > N$ and for every $t \in [a,b]$,

$$\left\| \int_a^t Y_n dX - \int_a^t Y_m dX \right\| < 3\epsilon. \quad (42)$$

Letting $m \to \infty$ in the above inequality, we obtain

$$\left\| \int_a^t Y_n dX - \int_a^t YdX \right\| < 3\epsilon. \quad (43)$$

It follows that for $n > N$,

$$\|Y_n - Y\|_{\mathcal{S}_X} < 3\epsilon. \quad (44)$$

Since $\epsilon > 0$ is arbitrary, this completes the proof.

VI. CONCLUSION

The lack of a satisfactory general and unified integration theory is still evident in mathematics, leaving many important goals in research only at best partially fulfilled. The extensions of both the vector integration theory and the stochastic integration theory in normed vector spaces given in this note go far beyond the classical treatment of the corresponding scalar cases. We have no doubt that our approach will open up new fields for researchers in functional analysis, in probability theory, in stochastic processes, and in broader areas of research.

REFERENCES

[1] J. Mass, "Malliavin calculus and decoupling inequalities in Banach spaces," *J. Mathematical Analysis and Applications*, vol. 363, no. 3, pp. 383-396, 2010

[2] S. Noreddine and J. Nourdin, "On the Gaussian approximation of vector-valued multiple integral," *J. Multivariate Analysis*, vol. 102, pp. 1008-1017, 2011

[3] A. S. Kussmaul, *Stochastic Integration and Generalized Martingales*, Pittman, London, 1977.

[4] J. Pellaumail, "Sur l'integrale stochastique et la décomposition de Doob-Meyer," *SMF*, vol. 9, 1973.

[5] N. Dinculeanu, V*ector Integration and Stochastic Integration in Banach Spaces*, J. Wiley & Sons, 2000.

[6] M. A. Robdera, "On strong and weak integrability of vector valued functions," *International J. Functional Analysis, Operator Theory and Application*, vol. 5, no. 1, pp. 63-81, 2013.

[7] M. A. Robdera, "Unified approach to vector valued integration," *International J. Functional Analysis, Operator Theory and Application*, vol. 5, no. 2, pp. 119-139, 2013.

[8] C. Dellacherie and P. A. Meyer, *Probabilités et Potentiel*, Hermann, Paris, 1975-1980.

[9] E. J. McShane, "Partial orderings and moore-smith limits," *Amer. Math. Monthly*, vol. 59, pp. 1-11, 1952.

[10] M. A. Robdera, "On differentiability of vector valued functions additive set functions," *Advance in Pure Mathematics*, vol. 3, pp. 653-659, 2013.

[11] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monographs in Mathematics, 2002.

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