Exact Controllability of Galerkin's Approximations for the Oldroyd Fluid System

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***Abstract*—In this paper we investigate the model of Oldroyd Fluid in a bounded smooth region of *Rn* (*n* = 2, 3) with control supported in a small subset of this domain. Under suitable assumptions on the Galerkin basis, we introduce Galerkin's approximations for the Oldroyd fluid system. Using the Hilbert Uniqueness Method in combination with the Schauder's fixed point, we prove the exact controllability for this finite-dimensional system.**

***Index Terms*—Exact controllability, oldroyd fluid, Galerkin'S approximations.**

I. Introduction

The focus of this paper is the study of the exact controllability of Galerkin's Approximations for the equation of the Oldroyd fluid. This model corresponding to an incompressible fluid, which is described by the following system of partial differential equations.

$$\begin{aligned}&\frac{\partial u}{\partial t}+u.\nabla u+\nabla p=div(\sigma)+F(x,t),\\&x\in\Omega,\ t>0,\ div(u)=0,\ x\in\Omega,\ t>0\end{aligned}\qquad(1)$$

With appropriate initial and boundary conditions. Here $\sigma = (\sigma_{ik})$ denotes the stress tensor with $tr\ \sigma = 0$, u represents the velocity vector, p is the pressure in the fluid and F is the external force. In fact, the stress tensor σ plays a special role, because the introduction of σ in (1) has the purpose of letting us consider reactions arising in the fluid during its motion. By establishing (Hookes's Law) the connection between σ and the tensor of deformation velocities $D=(D_{ik})=\frac{1}{2}\left(u_{ix_k}+u_{kx_i}\right)$, and their derivatives, we thus establish the type of fluid. Such relation between σ and D is called a *defining or r*heological equation or an equation of state (see [1]). From Newton's law, we set

$$\sigma = 2vD \qquad (2)$$

where is v the kinematic coefficient of viscosity, such fluid is called of Newtonian Fluid. Substituting (2) in (1) we obtain the equations of motion of Newtonian fluid, which is called *Navies-Stakes equations*.

Over the last century and half, the model of a Newtonian fluid has been the basic model of a viscous incompressible fluid. It describes flows of moderate velocities of the majority of viscous incompressible fluids encountered in practice. However, even eelier in the middle nineteenth century it was known that there exists viscous incompressible fluid not subject to the Newtonian equation (2). That is, it has a complex microstructure such as biological fluids, suspensions and liquid crystals, which are used in the current industrial process and shows (non-linear) viscoelastic behavior that cannot be described by the classical linear viscous Newtonian models. The first models of such fluids, were proposed in the nineteenth century by Maxwell [2] and [3], Kelvin [4], Voigt [5] and [6]. In the mid twentieth century Oldroyd extended such models [7]-[8].

The model of Oldroyd fluid (see [9], [7], [10]) can predict the stress relaxation as well as the retardation of deformation. Due to this, it has become popular for describing polymer suspension. In order to model the behavior of a dilute polymer solution in a Newtonian solvent, the extra stress tensor is often split in two components: a viscoelastic one and a purely viscous one. So the Oldroyd fluids of order one as it is known in the Russian literature (see [7], [11], [12], [13] and [14]) are described by defining relation.

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\sigma=2v\left(1+kv^{-1}\frac{\partial}{\partial t}\right)D \qquad (3)$$

where λ, v, k are positive constants with $(\nu - k\lambda - 1) > 0$. Here v denotes the kinematic viscosity, λ is the relaxation time, and k represents the retardation time. In the form of an integral equation, we write the above defining relation as.

$$\begin{aligned}&\sigma(x,t)=2k\lambda^{-1}D(x,t)+\\&2k\lambda^{-1}\left(v-k\lambda^{-1}\right)\int_0^t e^{\frac{-(t-\xi)}{\lambda}}D(x,t)d\xi\end{aligned}\qquad(4)$$

were $\sigma(x,0) = 0 = D(x,0)$.

Thus the equation of motion of the Oldroyd fluids of order one can be described most naturally by the system of integro-differential equations.

$$\begin{aligned}&\frac{\partial u}{\partial t}+(u\cdot\nabla)u-\mu\Delta u-\\&-\int_0^t\beta(t-\xi)\Delta u(x,\xi)\,d\xi+\nabla p=f,x\in\Omega,t>0,\end{aligned}\ (5)$$

And the incompressible condition

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$$div(u) = 0, x \in \Omega,\ t > 0 \quad (6)$$

With initial and boundary conditions

$u(x, 0) = u_0,\ x \in \Omega,\ and\ u(x, t) = 0\ x \in \Gamma,\ t \geq 0.$ (7)

Here, Ω is an open bounded connected set of $\mathbb{R}^n$ with smooth boundary Γ, $\mu = k\lambda^{-1}$ and kernel $g(t) = \gamma e^{-\delta t}$, where $\gamma = \lambda^{-1}(\nu - k\lambda^{-1})$ and $\delta = \lambda - 1$. For details of the physical background and its mathematical modeling, see [7], [9], [10] and [11].

As in Temam [15] let us denote Hm(Ω) the standard Hilbert-Sobolev space and by || . ||m the norm defined on it. When $m = 0$, we call H0 (Ω) as the space of square integrable functions $L^2(\Omega)$ with the usual norm | . | and inner product (., .). Further, let H_0^1 (Ω) be the completion of C_0^∞ (Ω) with respect to H_0^1 (Ω)-norm. We also use the following function spaces for the vector valued functions.

Let us consider the spaces:

$$V := \left\{ \varphi \in \left(C_0^\infty(\Omega) \right)^n : div(\varphi) = 0 \text{ in } \Omega, t > 0 \right\}$$

H: = the closure of V in $\left(L^2(\Omega)\right)^n$ and V:= the closure of V in $\left(H_0^1(\Omega)\right)^n$.

The spaces of vector functions are indicated by boldface, for instance, $\boldsymbol{H}_0^1(\Omega) = (H_0^1(\Omega))^n$ $\boldsymbol{L}^2(\Omega) = (L^2(\Omega))^n$. The inner products on $\boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{L}^2(\Omega)$ are denoted by

$$((\phi, \omega)) = \sum_{i=1}^{n} (\nabla \phi, \nabla \omega_i) \quad \text{and} \quad (\varphi, \omega) = \sum_{i=1}^{n} (\varphi_i, \omega_i)$$

respectively and norm by

$$\|\varphi\| = \left(\sum_{i=1}^{n} |\nabla \varphi_i|^2 \right)^{\frac{1}{2}} \text{ and } |\phi| = \left(\sum_{i=1}^{n} |\phi_i|^2 \right)^{\frac{1}{2}}$$

Respectively:

Note that under some smoothness assumptions on the boundary Γ, it is possible to characterize H and V as

$$H = \left\{ u \in \boldsymbol{L}^2(\Omega); div(u) = 0,\ u \cdot \eta\big|_\Gamma = 0 \right\}$$

$$V = \left\{ u \in \boldsymbol{H}_0^1(\Omega);\ div(u) = 0 \right\}$$

See Temam [15]. By Poincaré inequality, it can be shown that the norm of $\boldsymbol{H}_0^1(\Omega)$ is equivalent to $\boldsymbol{H}^1(\Omega) = \left(H^1(\Omega)\right)^n$ –norm. By V' we denote the dual of V.

Let $T > 0$ a real number. We denote by Q the cylinder $\Omega \times (0, T)$ of $\mathbb{R}^{n+1}$ with lateral boundary $\sum = \Gamma \times (0, T)$. Let be a Θ non-empty open subset of Ω. By we represent the characteristic functions on Θ. The main focus of this paper is to discuss the exact controllability for system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu\Delta u + (u \cdot \nabla)u - \displaystyle\int_0^1 g(t-s)\Delta u(s)ds + \nabla p = \mathrm{v}\chi_\Theta \ in \quad Q \\ div(u) = 0 \, in \quad Q \\ u = 0 \ on \quad \sum \\ u(x, 0) = u_0(x) \, on \quad \Omega \end{cases} \quad (8)$$

where $u(x\ t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the vector velocity (or state of the system) of fluid evaluated at the point (x, t), $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $p = p(x, t)$ is the pressure of the fluid evaluated at the point (x, t), μ represent a constant, $u_0(x)$ is the initial velocity and $g(t) = \gamma e^{-\delta t}$, where $\gamma = \lambda^{-1}(\nu - k\lambda^{-1})$ and $\delta = \lambda^{-1}$. The function $\nu = \nu(x, t)$ is the control function distributed in Θ.

In the context of the Navier-Stokes System, there are results of the local exact controllability to uncontrolled trajectories obtained in Fursikovlmanuvilov [16], lmanuvilov [17] and Femández-Cara *et al.*, [18]. The global approximate controllability of the 2-D Navier-Stokes equations with slip boundary conditions was obtained by Coron [19]. Combining results on global approximate controllability and local controllability, the global exact controllability for the Navier-Stokes system on a 2-D manifold was analyzed in Coron-Fursikov [20]. In [21] and [22], Lions and Zuazua introduced the finite-dimensional Galerkin's approximations for the Navier-Stokes system, and they proved that these Galerkin's approximations are exactly controllable and Araruna, Chaves-Silva and Rojas Medar [23] proved exact controllability of Galerkin's approximations of micropolar fluids. Optimal control problems associated with the Navier-Stokes equations also have a wide and important range in applications. This issue has been studied, for instance, by Fursikov in [24] and by Gunzburger and Hou in [25] and [26]. Concerning control results for the micropolar fluids, Femández-Cara and Guerrero in [27].

Our paper is organized as follows: In Section II, we state the basic notations. In Section III, we in traduce Galerkin's approximations for (1.1) and, for this finite dimensional system, we establish the exact controllability result. The proof is based on the Hilbert Uniqueness Method introduced by Lions (see, for instance, [28]) to study the exact controllability of linear systems and a fixed point technique.

II. GALERKIN'S APPROXIMATIONS

We represent the problem (1) in the variational formulation for all $\varphi \in V$,

$$\begin{cases} (u,\varphi) + \mu a(u,\varphi) + ((u \cdot \nabla)u, \varphi) + a(g * u, \varphi) = (\mathrm{v}\chi_\Theta, \varphi) \\ u(x,0) = u_0(x) in \quad \boldsymbol{V}, \end{cases} \quad (9)$$

For all $\varphi \in \boldsymbol{V}$

$$a(u, \mathrm{v}) = \sum_{i,j=1}^{3} \int \frac{\partial u_i}{\partial x_j} \frac{\partial \mathrm{v}_i}{\partial x_j} dx.$$

And $\left(g*u\right)\left(t\right)=\int_{0}^{t}g(t-s)u(s)ds.$

In V we consider a base $\left\{e_j\right\}_{j\geq 1}$ such that $\left\{e_j\right\}_{j\geq 1}$ are linearly independent in $\boldsymbol{L}^2(\Theta)$. (10)

The existence of this basis is guaranteed due to an abstract result proved in [19].

Let E be the finite-dimensional space

$$E=span\left[e_1,\ .....,e_n\right[\tag{11}$$

We introduce the Galerkin's approximations of the variation formulation (9) i.e.

$$\begin{vmatrix}(u_t,e)+\mu a(u,e)+\left((u\cdot\nabla)u,e\right)+a(g*u,e)=(\mathrm{v}\chi_{\Theta},e)\\ u(x,0)=u_0(x)in\quad E,\end{vmatrix} \tag{12}$$

For all $e\in E$

System (12) has a unique solution, $u\in C^0\left(\left[0,T\right];F\right)$ (see Apendix).

We say that (12) is exactly controllable in time T > 0 if, given $u_0,u^T\in E$, there exists a control $\mathrm{v}\in\boldsymbol{L}^2\left(\Theta\times\ (0,T)\right)$ such that the solution u of (12) satisfies

$$u(.,T;\mathrm{v})=u^T \tag{13}$$

We introduce the functional cost given by

$$J(u)=\frac{1}{2}\int_{\Theta\times(0,T)}\left|u\right|^2dxdt,u\in\boldsymbol{L}^2(\Theta\times(0,T) \tag{14}$$

n these conditions proof the following result:

Theorem 2.1. Let $T>0$ be given. Then the Galerkins approximations (12) is exactly controllable in the sense of (13). Furthermore, the functional cast of control given in (14) is bounded independently of the nonlinearity.

Proof. later on, to show and make explicit the cost of control can be bounded independently of the non- linearity, we introduce the family of systems

$$\begin{vmatrix}\frac{\partial u}{\partial t}-\mu\Delta u+(u\cdot\nabla)u-\int_0^1 g(t-s)\Delta u(s)ds+\nabla p=\mathrm{v}\chi_{\Theta}\ in\quad Q\\ div(u)=0\,in\quad Q\\ u=0\ on\ \sum\\ u(x,0)=u_0(x)\,on\quad\Omega\end{vmatrix} \tag{15}$$

where $\alpha\in R$. We will find some estimates independent of α.

We prove our main result for system (15). First, we introduce its variation formulation. Namely

$$\begin{vmatrix}(u_t,e)+\mu a(u,e)+\left((u\cdot\nabla)u,e\right)+\alpha(g*u,e)=(\mathrm{v}\chi_{\Theta},e)\\ u(x,0)=u_0(x)in\quad E,\end{vmatrix} \tag{16}$$

We proceed with this proof in several steps

Step 1 (Linear System). Take a function h such that

$$h\in L^2(0,T;E) \tag{17}$$

And we consider the linear system

$$\begin{vmatrix}(u_t,e)+\mu a(u,e)+\left((h\cdot\nabla)u,e\right)+\alpha(g*u,e)=(\mathrm{v}\chi_{\Theta},e)\\ u(x,0)=u_0(x)in\quad E,\end{vmatrix} \tag{18}$$

for all $e\in E$

System (18) has a unique solution $u\in C^0\left(\left[0,T\right];E\right)$, therefore due to the linearity of the problem we can be assume the data null. But, all results are valid as well if the initial data is not zero, that is, $u\left(0\right)=u_0\in E$.

Let us check that system (18) is exactly controllable in any time $T>0$, in the sense of (13). For this, it suffices to prove that if β ∈ E satisfies

$$(u(.,T;\mathrm{v}),\beta)=0,\forall\mathrm{v}\in L^2(\Theta\times(0,T))\Rightarrow\beta\equiv 0. \tag{19}$$

Note that the set $\left\{u\left(.,T,\mathrm{v}\right);\mathrm{v}\in\boldsymbol{L}^2\left(\Theta\times\ \left(0,T\right)\right)\right\}$ is a closed subspace of E.

Let us consider φ as being a solution of the ad- joint system

$$\begin{vmatrix}-(\varphi_t,e)+\mu a(\varphi,e)-\alpha\left((h\cdot\nabla)\varphi,e\right)+\\ +a\left(\int_t^T g(\eta-t)\varphi,e\right)+(\nabla\pi,e)=0)\\ \phi(x,T)=\beta\in E\end{vmatrix} \tag{20}$$

For all e ∈ E. System (20) has a unique solution $\varphi\in C^0\left(\left[0,T\right];E\right)$.

Taking $e=u\ (t)$ in (20), using the lemma 3.1 (see Appendix) and observing that

$$\left((h(t).\nabla)\varphi(t),u(t)\right)=-\left((h(t).\nabla)u(t),\varphi(t)\right),\forall t\in\left[0,T\right],$$

We get, after integration by parts in t, that

$$\begin{aligned}&-(u(T),\varphi(T))+\int_0^T[(u_t,\varphi)+\mu a(u,\varphi)]dt+\\ &\int_0^T[\alpha(h\cdot\nabla)u,\varphi)+a(g*u,\varphi)+(\nabla\pi,u)]dt=0\end{aligned} \tag{21}$$

By (18) we obtain

$$-(u(T),\beta)=\int_0^T(\mathrm{v}\chi_{\Theta},\varphi)dt \tag{22}$$

Hence,

$$0=\int_0^T(\mathrm{v}\chi_{\Theta},\varphi)dt,\forall\mathrm{v}\in L^2(\Theta\times(0,T) \tag{23}$$

So, one assures

$$\varphi\equiv 0\ \text{ in }\ \Theta\times(0,T). \tag{24}$$

Since $\varphi\equiv\sum_{t=1}^{N}\varphi i(t)ei$ and thanks to (10), we can guarantee by (24) that $\varphi i\ (t)=0$, for $i=1,...,N$. Thus,

$\varphi \equiv 0$ and, therefore $\beta \equiv 0$. Thus, system (18) is exactly controllable.

Step 2 (Estimates). Thanks to the results obtained in Step 1, one can define the functional $M : L^2(0,T;F) \to \mathbb{R}$ by

$$M(h) = \inf_{v \in Uad} \frac{1}{2} \int_{\Theta\times(0,T)} |v|^2 \, dxdt \tag{25}$$

where U_{ad} is the set of admissible controls $U_{ad} = \{v \in \boldsymbol{L}^2(\Theta \times (0,T))\}$, u solution of (18)}, satisfying (13). Note that U_{ad} is a closed convex of $\boldsymbol{L}^2(\Theta\times(0,T))$ and $J(v)$ is convex $s.c.i$ and coercive.

We are interested in proving that

$$M(h) \le \{\text{constant independent of } h \text{ and } \alpha\} \tag{26}$$

For this, we use a duality argument.

We consider the continuous linear operator $L : \boldsymbol{L}^2(\Theta\times(0,T)) \to F$ defined by $L(V) = u(., T; V)$ and we introduce the functional.

$$F_1(v) = \frac{1}{2} \int_{\Theta\times(0,\,T)} |v|^2 dxdt$$

And

$$F_2(B) = \begin{cases} 0, \text{if } \beta = u^T \\ \infty, \text{otherwise} \end{cases}$$

In this way, we can rewrite the functional M as follows:

$$M(h) = \inf_{v\in \boldsymbol{L}^2(\Theta\times(0,T))} [F_1(v) + F_2(L(v))] \tag{27}$$

From the duality theorem of Frenchel and Rockefeller, see [29] we have.

$$-M(h) = \inf_{\beta\in E} [F_1^*(L^*(\beta) + F_2^*(-\beta)] \tag{28}$$

where $L^* E \to \boldsymbol{L}^2(\Theta\times(0,T))$ is the adjoin of L.

Using (22), one sees that

$$L^*(\beta) = -\varphi \text{ in } \Theta\times(0,\mathrm{T}). \tag{29}$$

Since

$$F_1^*(\varphi) = \frac{1}{2} \int_{\Theta\times(0,\,T)} |\varphi|^2 dxdt$$

And

$$F_2^*(-\beta) = -(\beta, u^T),$$

Then

$$-M(h) = \inf_{\beta\in E} \left[\frac{1}{2} \int_{\Theta\times(0,T)} |\varphi|^2 dxdt - (\beta, u^T)\right] \tag{30}$$

As E is a subspace of finite dimension, it follow that the three norms $|\beta|$, $\|\beta\|$ and $\left(\int_\Theta |\beta|^2 dx\right)^{\frac{1}{2}}$ are equivalent on E, on that

$$C|e|^2 \ge \int_\Theta |e|^2 \, dx \ge c\|e\|^2, \forall e \in E$$

with C and C positive constants that only depend on E, where

$$|e|^2 = \int_\Omega |e|^2 \, dx, \forall e \in E.$$

Thus, (28) gives

$$-M(h) \ge \inf_{\beta\in E} \left[\frac{c}{2} \int_{\Theta\times(0,T)} |\varphi|^2 dxdt - (\beta, u^T)\right] \tag{31}$$

Now, we take $e = \phi\ (t)$ in (20) and integrating from t to T. Then the terms containing h drop out and we obtain

$$\frac{1}{2}|\varphi(t)|^2 + \int_t^T \mu a(\varphi,\varphi) ds + \int_t^T a(\int_t^T g(\eta - t)\varphi, \varphi) ds = \frac{1}{2}|\beta|^2.$$

Integrating in $(0, T)$ we get

$$\frac{1}{2}\int_0^T |\varphi(t)|^2 dt + \int_0^T ta(\varphi,\varphi) dt + \int_0^T ta\left(\int_0^T g(\eta - t)\varphi, \varphi\right) ds = \frac{T}{2}|\beta|^2 \tag{32}$$

Notice that $a(\varphi,\varphi) = \|\varphi\|^2 \le C|\varphi|^2$ and $a\left(\int_t^T g(\eta - t)\varphi, \varphi\right) \le \|g\|_\infty \|\varphi\|^2$ for some $C > 0$ that depends only on E because E are finite dimensional.

So

$$\frac{T}{2}|\beta|^2 \le (\frac{1}{2} + CT + T\|g\|_\infty) \int_0^T \|\varphi(t)\|^2 \tag{33}$$

For suitable $C > 0$ that depends only on E. In view of (31) we get

$$-M(h) \ge \inf_{\beta\in E} \left[k\|\beta\|^2 - (\beta, u^T)\right] \tag{34}$$

where

$$k = \frac{cT}{2 + 4T + 4T\|g\|_\infty} \text{ Hence,} \tag{35}$$

Which implies (26).

Step 3 (Nonlinear system) Let $h \in L^2(0,T;E)$ be given. We choose for v the unique element such that

$$\frac{1}{2}\int_{\Theta\times(0,\,T)} |v|^2 dxdt = M(h). \tag{36}$$

We define a continuous mapping $h \to v$ *from* $L^2(0,T;E)$ in $\boldsymbol{L}^2(\Theta\times(0,T))$. We denote by $u(h)$ the solution of (18) with the control v=$u(h)$. Taking $e = u(t)$ in (18) we obtain.

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \mu a(u(t), u(t)) + a((g * u)(t), u(t) = (v\chi_\Theta, u(t)).$$

Thus, integrating the last equation in (0, *t*) we obtain

$$\frac{1}{2}\|u(t)\|^2+\int_0^t \mu a\big(u(s),u(s)\big)ds+\int_0^t \big(a(g*u)(s),u(s)\big)\,ds \le \|v\|_{\mathbf{L}^2(\Theta\times(0,T))}\|u\|u(h)_{\mathbf{L}^2(\Theta\times(0,T))} \quad (37)$$

By lemmas 3.1 and 3.2 we have

$$\|u(t)\|^2 \le \|v\|^2_{\mathbf{L}^2(\Theta\times(0,T))}\|u\|^2_{\mathbf{L}^2(\Theta\times(0,T))} \quad (38)$$

From Gronwall's inequality, we have by (38) that

$$\|u(t)\|^2 \le CT\|v\|^2_{L^2(\Theta\times(0,T))} \quad (39)$$

In view of (26) we obtain that, when *h* varies in $L^2(0,T;E)$

$$u \text{ remains a bounded subset } K_1 \subset L^2(0,T;E). \quad (40)$$

We claim that

$$\text{the map } h\to u(h) \text{ admits a fixed point in } K_1. \quad (41)$$

In fact, according to Schauder's fixed point theorem, it suffices to show that the range of u (*h*) when *h* spans *K*1 is relatively compact in *K*1, which follows from the following estimate:

$$u_t \text{ remains in a bounded set } L^2(0,T;E) \text{ when h describes } K_1. \quad (42)$$

To prove (42) we observe that, from (18), the following holds

$$|(u_t,e)| \le C\big(\alpha\|h(t)\|\|\nabla u(t)\|+\mu\|\nabla u(t)\|\big)\|e\| + C(\|\mathrm{v}\|_{L^2(\Theta)}+\|g\|_\infty\|\nabla u(t)\|)\|e\|, \forall e\in E \quad (43)$$

Since on the finite-dimensional space *E* all the norms are equivalent. Therefore

$$\|u_t\| \le C\big(\alpha\|h(t)\|\|\nabla u(t)\|+\mu\|\nabla u(t)\|\big)+ +C\Big(\|\mathrm{v}\|_{L^2(\Theta)}+\|g\|_\infty\|\nabla u(t)\|\Big), \forall e\in E, \quad (44)$$

Which implies 42.

APPENDIX

In this section we will show any technical lemmas necessary to show the existence and uniqueness of solutions.

Lemma 3.1. Let g: [0, ∞) → [0, ∞) be a function of *L*1(0,∞) and y, $\xi\in L^2(;T;L^2(\Omega))$ Then

$$\int_{\Omega\times(0,T)}\left[\int_0^t g(t-s)y(s)ds\right]\xi(t)dt = \int_{\Omega\times(0,T)}\left[\int_0^t g(\eta-s)\xi(s)d\eta\right]y(t)dt$$

Proof. Consider

$$\hat{y} = \begin{vmatrix} y \text{ in } [0,T] \\ 0 \text{ out side of } [0,T]\end{vmatrix} \qquad \tilde{\varsigma} = \begin{vmatrix} \varsigma \text{ in } [0,T] \\ 0 \text{ out side of } [0,T]\end{vmatrix}$$

$$\hat{g} = \begin{vmatrix} g(s) \text{ if } s\ge 0 \\ 0 \text{ if } s<0\end{vmatrix} \quad\text{and}\quad \widetilde{g*y} = \begin{vmatrix} g*y \text{ in } [0,T] \\ 0 \text{ out side } of\,[0,T].\end{vmatrix}$$

Thus $\tilde{y}\in L^2(\mathbb{R};L^2(\Omega))$, $\tilde{\varsigma}\in L^2(\mathbb{R};L^2(\Omega))$ and g $\tilde{g}\in L^1(\mathbb{R})$ By Lemma 2.1 it follows that $g*y\in L^2(\mathbb{R};H)$ Therefore,

$$\begin{aligned}
&\int_{\Omega\times(0;T)}\int_0^t [g(t-s)y(s)ds]\varsigma(t)dt = \\
&\int_{\Omega\times(0;T)} g*y(t)\varsigma(t)dt = \\
&\int_{\Omega\times IR} \widetilde{g*y(t)}\tilde{\varsigma}(t)dt = \\
&\int_{\Omega\times IR} \tilde{g}*\tilde{y}(t)\tilde{\varsigma}(t)dt := \\
&\int_{\Omega\times IR} \tilde{y}(t)\breve{g}*\tilde{\varsigma}(t)dt = \\
&\int_{\Omega\times IR}\left[\int_{IR}[\tilde{g}(t-\eta)\tilde{\varsigma}(\eta)d\eta\right]\tilde{y}(t)dt = \\
&\int_{\Omega\times IR}\left[\int_{IR}\tilde{g}(\eta-t)\tilde{\varsigma}(\eta)d\eta\right]\tilde{y}(t)dt = \\
&\int_{\Omega\times(0,T)}\int_t^T [g(\eta-t)\varsigma(\eta)d(\eta)]y(t)d(t),
\end{aligned}$$

where by $\breve{g}$ we denote $\breve{\tilde{g}}(x)=\tilde{g}(-x)$.

Lemma 3.2. For an arbitrary α, *T* ∈ ℝ with $\alpha>0$, $T>0$ and $h\in L^2(0,T;L^2(\Omega))=L^2(\Omega)$ we have

$$I := \int_{\Omega\times(0,T)}\left[\int_0^t e^{-a(t-s)}h(x,s)ds\right]h(x,t)dtdx \ge 0$$

Proof. From Lemma 3.1 we obtain

$$I = \int_{\Omega\times(0,T)}\left[\int_0^T e^{-a(t-s)}h(x,\eta)d\eta\right]h(x,t)dtdx$$

Hence,

$$I = \frac{1}{2}\int_{\Omega\times(0,T)}\left[\int_0^T e^{-\alpha|t-\eta|_{IR}}h(x,\eta)d\eta\right]h(x,t)dtdx$$

We note that form this identity the lemma is true for the case $\alpha=0$. On the other hand, for $\alpha=0$, let $h\times(x,t)=h(x,t)$ in $\Omega\times(0,T)$ and $h\times(x,t)=0$ outside of $\Omega\times(0,T)$. Then, from identity above it follows that

$$I = \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}}\left[\int_{\mathbb{R}} e^{-\alpha|t-s|}\,h*(x,\eta)d\eta\right]h*(x,t)dtdx$$

We make the change of variables $t-\eta=\xi$ to obtain

$$I = \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}}\left[\int_{\mathbb{R}} e^{-\alpha|\xi|_{IR}}h*(x,t-\xi)d\xi\right]h*(x,t)dtdx$$

The key idea to use the utilize Fourier Transform of $h\times(x,t)$. In fact,

$$I = \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}^n \times \mathbb{R}} h^*(x,t) \times \int_{\mathbb{R}} e^{-\alpha|\xi|_{IR}} \times$$

$$\left\{ \int_{\mathbb{R}^n \times \mathbb{R}} e^{i(x,t-\xi)\bullet(y,\eta)} \hat{h}^*(y,\eta) d\eta dy \right\} dtdx =$$

$$\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}^n \times \mathbb{R}} h^*(x,t) \times \int_{\mathbb{R}^n \times \mathbb{R}} e^{i(x,t)\bullet(y,\eta)} \hat{h}^*(y,\eta) \times$$

$$\left\{ \int_{\mathbb{R}} e^{-i\xi\eta} e^{-\alpha|\xi|} d\xi \right\} d\eta dy dt dx =$$

$$\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}^n \times \mathbb{R}} h^*(x,t)$$

$$\left[\int_{\mathbb{R}^n \times \mathbb{R}} e^{i(x,t)\bullet(y,\eta)} \hat{h}^*(y,\eta) \widehat{e^{-\alpha|\eta|_{IR}}} d\eta dy \right] dtdx =$$

$$\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}^n \times \mathbb{R}} \widehat{e^{-\alpha|\eta|_{\mathbb{R}}}} \hat{h}^*(y,\eta)$$

$$\left[\int_{\mathbb{R}^n \times \mathbb{R}} e^{i(x,t)\bullet(y,\eta)} h^*(x,t) dtdx \right] d\eta dy =$$

$$\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}^n \times \mathbb{R}} \widehat{e^{-\alpha|\eta|_{\mathbb{R}}}} \hat{h}^*(y,\eta) \quad -$$

$$\left[\int_{\mathbb{R}^n \times \mathbb{R}} e^{i(x,t)\bullet(y,\eta)} \overline{h^*(x,t)} dtdx \right] d\eta dy =$$

$$\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}^n \times \mathbb{R}} \widetilde{e^{-\alpha|\eta|_{\mathbb{R}}}} \hat{h}^*(y,\eta) \times$$

$$\overline{\left[\int_{\mathbb{R}^n \times \mathbb{R}} e^{i(x,t)\bullet(y,\eta)} h^*(x,t) dtdx \right]} d\eta dy =$$

$$\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}^n \times \mathbb{R}} \widehat{e^{-\alpha|\eta|_{\mathbb{R}}}} \left| \overline{\hat{h}}^*(y,\eta) \right|_{\mathbb{R}}^2 d\eta dy.$$

$$\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}^n \times \mathbb{R}} \widehat{e^{-\alpha|\eta|_{\mathbb{R}}}} \left| \overline{\hat{h}}^*(y,\eta) \right|_{\mathbb{R}}^2 d\eta dy.$$

Therefore

$$I = \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}^n \times \mathbb{R}} \widehat{e^{-\alpha|\eta|_{\mathbb{R}}}} \left| \overline{\hat{h}}^*(y,\eta) \right|_{\mathbb{R}}^2 d\eta dy.$$

Recalling the well know formula

$$\widehat{e^{-\alpha|\eta|_{\mathbb{R}}}} = \frac{2\alpha}{\sqrt{2\pi}(\alpha^2 + \eta^2)}$$

We have that

$$I = \frac{\alpha}{2\pi} \int_{\mathbb{R}^n \times \mathbb{R}} (a^2 + \eta^2)^{-1} \left| \hat{h}^* \times (y,\eta) \right|_{\mathbb{R}}^2 d\eta dy \geq 0$$

where by $h*$ and $h*$ we denote the Fourier Transform and the conjugate of $h*$ respectively. This closes the proof of the Lemma.

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