Modified Trial Equation Method to the Nonlinear Fractional Sharma–Tasso–Olever Equation

Hasan Bulut and Yusuf Pandir

***Abstract*—In this paper, we apply the modified trial equation method to fractional partial differential equations. The fractional partial differential equation can be converted into the nonlinear non-fractional ordinary differential equation by the fractional derivative and traveling wave transformation. So, we get some traveling wave solutions to the time-fractional Sharma–Tasso–Olever (STO) equation by the using of the complete discrimination system for polynomial method. The acquired results can be demoted by the soliton solutions, single-king solution, rational function solutions and periodic solutions.**

***Index Terms*—The modified trial equation method, fractional Sharma–Tasso–Olever equation, soliton solution, periodic solutions.**

I. Introduction

In recent years, the fractional differential equations play an important role in various applications in physics, biology, engineering and control theory. The nonlinear fractional partial differential equations represent the mathematical modelling of various real life problems. In order to solve these problems, a general method cannot be defined even in the most useful works. Also, a remarkable progress has been become in the construction of the approximate solutions for fractional nonlinear partial differential equations [1]-[3]. Several powerful methods have been proposed to obtain approximate and exact solutions of fractional differential equations, such as the Adomian decomposition method [4], [5], the homotopy analysis method [6], [7], the homotopy perturbation method [8], and so on. The exact solutions of these problems, when they exist, are very important in the understanding of the nonlinear fractional physical phenomena.

Liu introduced a new approach called the complete discrimination system for a polynomial to classify the traveling wave solutions as nonlinear evolution equations and applied this idea to some nonlinear partial differential equations [9]-[11]. Furthermore, some authors [12], [13] used the trial equation method proposed by Liu. However, we established a new trial equation method to obtain 1-soliton, singular soliton, elliptic integral function and Jacobi elliptic function solutions or the others to nonlinear partial differential equations with generalized evolution in [14]-[16].

In Section II, we give some useful definitions and properties of the fractional calculus and also produce a modified trial equation method for fractional nonlinear evolution equations.

In Section III, as an application, we solve the nonlinear fractional partial differential equation such as the time-fractional Sharma–Tasso–Olever equation [17], [18]

$$\frac{\partial^\alpha u}{\partial t^\alpha}+3au_x^2+3au^2u_x+3auu_{xx}+au_{xxx}=0 \qquad (1)$$

where a is arbitrary constants and α is a parameter describing the order of the fractional time-derivative.

In this research, we obtain the classification of the wave solutions to Eq. (1), and derive some new solutions. Using the modified trial equation method, we find some new exact solutions of the fractional nonlinear physical problem. The purpose of this paper is to obtain exact solutions of nonlinear fractional Sharma–Tasso–Olever equation by modified trial equation method.

II. Preliminaries

In this section of the paper, it would be helpful to give some definitions and properties of the modified Riemann-Liouville derivative. For an introduction to the classical fractional calculus we indicate the reader to [1]-[3]. Here, we shortly review the modified Riemann-Liouville derivative from the recent fractional calculus proposed by Jumarie [19]-[21]. Let $f:[0,1]\to\Re$ be a continuous function and $\alpha\in(0,1)$. The Jumarie modified fractional derivative of order α and f may be defined by expression of [22] as follows:

$$D_x^\alpha f(x)=\begin{cases}\dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^x (x-\xi)^{-\alpha-1}[f(\xi)-f(0)]d\xi, & \alpha<0;\\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)]d\xi, & 0<\alpha<1;\\ \left(f^{(n)}(\xi)\right)^{\alpha-n} & , n\le\alpha\le n+1, n\ge 1\end{cases} \qquad (2)$$

In addition to this expression, we may give a summary of the fractional modified Riemann-Liouville derivative properties as follows:

$$D_x^\alpha k=0$$

$$D_x^\alpha x^\mu=\begin{cases}0, & \mu\le\alpha-1\\ \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}x^{\mu-x}, & \mu>\alpha-1\end{cases} \qquad (3)$$

$$D_x^\alpha\left[f(u(x))\right]=f_u'(u)D_x^\alpha u(x)=D_x^\alpha f(u)(u_x')^\alpha$$

In this paper, a new approach to the trial equation method will be given. In order to apply this method to fractional nonlinear partial differential equations, we consider the

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Hasan Bulut is with Mathematics Dept., Firat University, Elazig, 23119 Turkey (e-mail: hbulut@firat.edu.tr).

Yusuf Pandir is with Mathematics Dept., Bozok University, Yozgat, 66100 Turkey (e-mail: yusufpandir@ gmail.com).

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following steps.

Step 1. We consider time fractional partial differential equation in two variables and a dependent variable u

$$P(u, D_t^\alpha u, u_x, u_{xx}, u_{xxx}, \cdots) = 0 \quad (4)$$

and take the wave transformation

$$u(x,t) = u(\eta), \qquad \eta = kx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \quad (5)$$

where $\lambda \neq 0$. Substituting Eq. (5) into Eq. (4) yields a nonlinear ordinary differential equation

$$N(u, u', u'', u''', \cdots) = 0 \quad (6)$$

Step 2. Take trial equation as follows:

$$u' = \frac{F(u)}{G(u)} = \frac{\sum_{i=0}^{n} a_i u^i}{\sum_{j=0}^{l} b_j u^j}, \quad (7)$$

and

$$u'' = \frac{F(u)(F'(u)G(u) - F(u)G'(u))}{G^3(u)} \quad (8)$$

where $F(u)$ and $G(u)$ are polynomials. Substituting above relations into Eq. (6) yields an equation of polynomial $\Omega(u)$ of u:

$$\Omega(u) = \rho_s u^s + \cdots + \rho_1 u + \rho_0 = 0 \quad (9)$$

According to the balance principle, we can get a relation of n and l. We can compute some values of n and l.

Step 3. Let the coefficients of $\Omega(u)$ all be zero will yield an algebraic equations system:

$$\rho_i = 0, \qquad i = 0, \cdots, s \quad (10)$$

Solving this system, we will specify the values of $a_0, \cdots, a_n$ and $b_0, \cdots, b_l$.

Step 4. Reduce Eq. (7) to the elementary integral form

$$\pm(\mu - \mu_0) = \int \frac{G(u)}{F(u)} du \quad (11)$$

Using a complete discrimination system for polynomial to classify the roots of $F(u)$, we solve Eq. (11) with the help of MATHEMATICA and classify the exact solutions to Eq. (6). In addition, we can write the exact traveling wave solutions to Eq. (4), respectively.

III. Application to the Sharma-Tasso-Olever Equation

In this section, we apply the method developed in Section 2 to the nonlinear fractional Sharma–Tasso–Olever equation. In the case of $\alpha = 1$, Eq. (1) reduces to the classical nonlinear Sharma–Tasso–Olever equation. Many researchers have tried to get the exact solutions of this equation by using a variety of methods [23]-[25].

Let us consider the travelling wave solutions of Eq. (1), and we perform the transformation $u(x,t) = u(\eta)$, $\eta = kx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}$ where k, λ are constants. Then, integrating this equation with respect to η and setting the integration constant to zero, we get

$$-\lambda u + ak^3 u'' + aku^3 + 3ak^2 u u' = 0 \quad (12)$$

Substituting, Eqs. (7) and (8) into Eq. (12) and using balance principle yields

$$n = l + 2$$

This resolution procedure is applied and we obtain results as follows:

Case 1:

If we take $l = 0$, then $n = 2$,

$$u' = \frac{a_0 + a_1 u + a_2 u^2}{b_0},$$

$$u'' = \frac{(a_1 + 2a_2 u)(a_0 + a_1 u + a_2 u^2)}{b_0^2} \quad (13)$$

where $a_2 \neq 0$, $b_0 \neq 0$. Thus, we have a system of algebraic equations from the coefficients of polynomial of u. Solving the algebraic equation system (10) yields the following:

Case 1.1

$$a_0 = 0,\ a_1 = a_1,\ a_2 = -\frac{b_0}{k},\ b_0 = b_0,\ \lambda = \frac{ak^3 a_1^2}{b_0^2} \quad (14)$$

Substituting these coefficients into Eq. (6) and (11), we have

$$\pm(\mu - \mu_0) = \int \frac{b_0}{a_2 u^2 + a_1 u} du \quad (15)$$

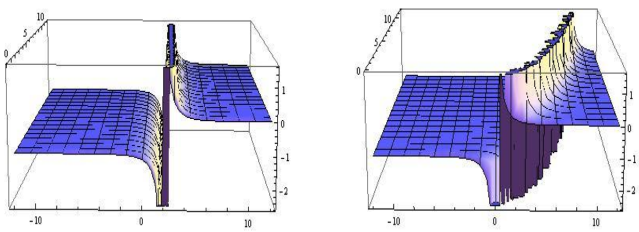
Integrating eq. (15), we procure the solution to the Eq. (1) as follows:

$$u(x,t) = \frac{a_1}{\exp\left[\pm \frac{a_1}{b_0}\left(kx - \frac{a k^3 a_1^2 t^\alpha}{b_0^2 \Gamma(1+\alpha)} - \eta_0\right)\right] - a_2}. \quad (16)$$

If we take $\eta_0 = 0$ and $a_1 = a_2 = 1$, then the solutions (16) can reduce to single king solution,

$$u(x,t) = \frac{1}{\exp(B(x - \lambda_1 t^\alpha)) - 1} \quad (17)$$

where $B = \pm \frac{k}{b_0}$, $\lambda_1 = \frac{a k^2}{b_0^2 \Gamma(1+\alpha)}$. Here, B is the inverse width of the solitons.



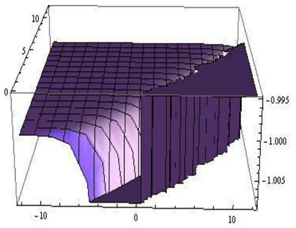


Fig. 1. Graph of the solution (17) corresponding to the values $\alpha = 0.01,\ 0.5$ and $\alpha = 0.85$ from left to right when $k = b_0 = 1,$ and $a = 2.25.$

Case 1.2:

$$a_0 = a_0,\ a_1 = 0,\ a_2 = -\frac{b_0}{k},\ b_0 = b_0,\ \lambda = \frac{ak^2 a_0}{b_0} \quad (18)$$

Substituting these coefficients into Eq. (6) and (11), we have

$$\pm(\mu - \mu_0) = \int \frac{b_0}{a_2 u^2 + a_0} du \quad (19)$$

Integrating eq. (19), we procure the solution to the Eq. (1) as follows:

$$u(x,t) = \sqrt{\frac{a_0}{a_2}} \tan\left[\pm\frac{\sqrt{a_0 a_2}}{b_0}\left(kx - \frac{a k^2 a_0 t^\alpha}{b_0 \Gamma(1+\alpha)} - \eta_0\right)\right] \quad (20)$$

If we take $\eta_0 = 0$, then the solutions (20) can reduce to periodic solution,

$$u(x,t) = M \tan\left(B\left(x - \lambda_2 t^\alpha\right)\right) \quad (21)$$

where $M = \sqrt{\frac{a_0}{a_2}}$, $B = \pm\frac{k\sqrt{a_0 a_2}}{b_0}$ $\lambda_2 = \frac{a k^2 a_0}{b_0 \Gamma(1+\alpha)}$. Here, B is the inverse width of the solitons.

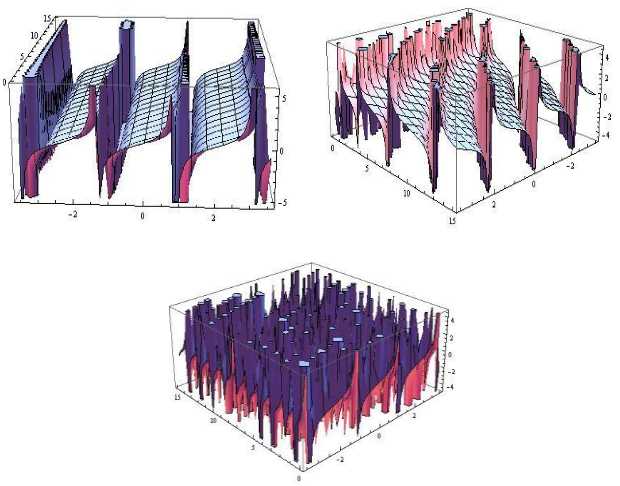


Fig. 2. Graph of the solution (21) corresponding to the values $\alpha = 0.01,\ 0.5$ and $\alpha = 0.85$ from left to right when $k = a_0 = b_0 = 1,\ a_2 = 2$ and $a = 2.25$

Remark 1. If we let the corresponding values for some parameters, solution (21) is in full agree with the solution (53) mentioned in Ref. [17].

Remark 2. The solutions (17) and (21) obtained by using the modified trial equation method for Eq. (1) have been checked by Mathematica. To our knowledge the rational function solution and periodic solution that we find in this paper are new traveling wave solutions of Eq. (1).

Case 2:

If we take $l = 1$ and $n = 3$, then

$$u' = \frac{a_0 + a_1 u + a_2 u^2 + a_3 u^3}{b_0 + b_1 u}, \quad (22)$$

$$u'' = \frac{(a_0 + a_1 u + a_2 u^2 + a_3 u^3)\left((b_0 + b_1 u)(a_1 + 2a_2 u + 3a_3 u^2) - b_1(a_0 + a_1 u + a_2 u^2 + a_3 u^3)\right)}{(b_0 + b_1 u)^3}, \quad (23)$$

where $a_3 \neq 0$, $b_1 \neq 0$. Respectively, solving the algebraic equation system (10) yields the following:

Case 2.1:

$$a_0 = \frac{a_1 b_0}{b_1},\ a_1 = a_1,\ a_2 = -\frac{b_0}{k},\ a_3 = -\frac{b_1}{k},$$

$$b_0 = b_0,\ b_1 = b_1,\ \lambda = \frac{ak^2 a_1}{b_1}. \quad (24)$$

Substituting these coefficients into Eq. (6) and (11), we have

$$\pm(\mu - \mu_0) = \int \frac{b_0 + b_1 u}{a_3 u^3 + a_2 u^2 + a_1 u + a_0} du \quad (25)$$

Integrating eq. (25), we procure the solution to the Eq. (1) as follows:

$$\pm(\mu - \mu_0) = -\frac{b_0 - b_1\alpha_1 + 2b_1 u}{2(u - \alpha_1)^2}, \quad (26)$$

$$\pm(\mu - \mu_0) = \frac{(b_0 + b_1\alpha_2)\ln\left|\frac{u-\alpha_2}{u-\alpha_1}\right| - \frac{(b_0 + b_1\alpha_1)(\alpha_1 - \alpha_2)}{u - \alpha_1}}{(\alpha_1 - \alpha_2)^2}, \quad (27)$$

$$\pm(\mu - \mu_0) = \frac{(b_0 + b_1\alpha_1)\ln|u-\alpha_1|}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)} - \frac{(b_0 + b_1\alpha_2)\ln|u-\alpha_2|}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)} + \frac{(b_0 + b_1\alpha_3)\ln|u-\alpha_3|}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)}. \quad (28)$$

Also α_1, α_2 and α_3 are the roots of the polynomial equation

$$\Gamma^3 + \frac{a_2}{a_3}\Gamma^2 + \frac{a_1}{a_3}\Gamma + \frac{a_0}{a_3} = 0. \quad (29)$$

Substituting the solutions (26) into (11), then we find solution

$$u(x,t) = \frac{b_1 + 2\alpha_1(\eta - \eta_0) \pm \sqrt{b_1^2 - (2b_0 2b_1\alpha_1)\eta + 2(b_0 + b_1\alpha_1)\eta_0}}{2(\eta - \eta_0)}. \quad (30)$$

If we take $\eta_0 = 0$, then the solution (30) can reduce to rational function solution

$$u(x,t) = \frac{b_1 + 2\alpha_1\left(kx - \frac{ak^2 a_1 t^\alpha}{b_1\Gamma(1+\alpha)}\right) \pm \sqrt{b_1^2 - (2b_0 2b_1\alpha_1)\left(kx - \frac{ak^2 a_1 t^\alpha}{b_1\Gamma(1+\alpha)}\right)}}{2\left(kx - \frac{ak^2 a_1 t^\alpha}{b_1\Gamma(1+\alpha)}\right)}. \quad (31)$$

For simplicity we rewrite for the solution (31) as follows:

$$u(x,t) = \frac{b_1 + B_1\left(x - \lambda_3 t^\alpha\right) \pm \sqrt{b_1^2 - B_2\left(x - \lambda_3 t^\alpha\right)}}{2k\left(x - \lambda_3 t^\alpha\right)} \quad (32)$$

where $B_1 = 2\alpha_1 k,\ B_2 = (2b_0 + 2b_1\alpha_1)k,\ \lambda_3 = \dfrac{a\,ka_1}{b_1\Gamma(1+\alpha)}$.

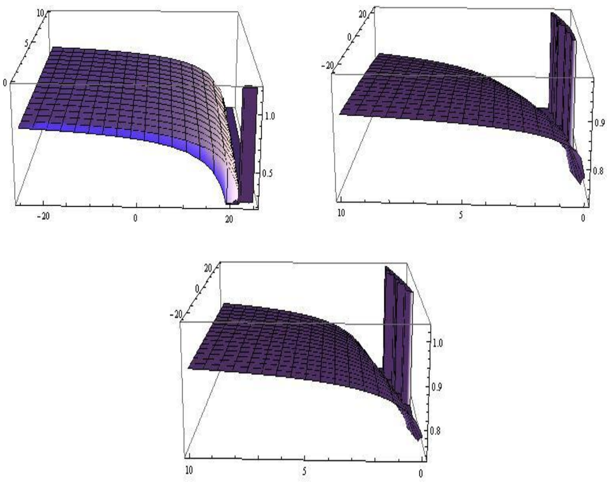


Fig. 3. Graph of the solution (32) corresponding to the values $\alpha = 0.01,\ 0.5$ and $\alpha = 0.85$ from left to right when $k = a_1 = \alpha_1 = b_0 = b_1 = 1$, and $a = 25.25$

Remark 3. The solutions (26)–(28) computed in case 2.1 have been checked by Mathematica. We think that these solutions have not been found in the literature of Eq. (1).

Case 2.2:

$$a_0 = 0,\ a_1 = \frac{b_0(ka_2 + b_0)}{kb_1},\ a_2 = a_2,\ a_3 = -\frac{b_1}{k},$$

$$b_0 = b_0,\ b_1 = b_1,\ \lambda = \frac{ak(ka_2 + b_0)^2}{b_1^2} \quad (33)$$

Substituting these coefficients into Eq. (6) and (11), we have

$$\pm(\mu - \mu_0) = \int \frac{b_0 + b_1 u}{a_3 u^3 + a_2 u^2 + a_1 u}\,du \quad (34)$$

Integrating eq. (34), we procure the solution to the Eq. (1) as follows:

$$\pm(\mu - \mu_0) = \frac{2(a_2 b_0 - a_1 b_1)\,Arc\tan\left(\frac{a_2 + 2a_3 u}{N}\right) + b_0\, N \ln\left|\frac{a_1 + u(a_2 + a_3 u)}{u^2}\right|}{2a_1 N}, \quad (35)$$

where $N = \sqrt{4a_1 a_3 - a_2^2}$.

Case 2.3:

$$a_0 = a_2 = 0,\ a_1 = \frac{b_0^2}{kb_1},\ \ a_3 = -\frac{b_1}{k},$$

$$b_0 = b_0,\ b_1 = b_1,\ \lambda = \frac{akb_0^2}{b_1^2} \quad (36)$$

Substituting these coefficients into Eq. (6) and (11), we have

$$\pm(\mu - \mu_0) = \int \frac{b_0 + b_1 u}{a_3 u^3 + a_1 u}\,du \quad (37)$$

Integrating eq. (37), we procure the solution to the Eq. (1) as follows:

$$\pm(\mu - \mu_0) = \frac{2b_1\sqrt{a_1}\,Arc\tan\left(\sqrt{\frac{a_3}{a_1}}u\right) + b_0\sqrt{a_3}\ln\left|\frac{u^2}{a_1 + a_3 u^2}\right|}{2a_1\sqrt{a_3}} \quad (38)$$

For a better understanding, we plot solutions (17), (21) and (32) of the nonlinear fractional Sharma–Tasso–Olever equation in Fig. 1-3, which shows the dynamics of solutions with suitable parametric choices.

IV. Conclusions

In this paper, the modified trial equation method is studied for the nonlinear fractional differential equations. We used it to obtain some soliton and rational function solutions to the time-fractional nonlinear Sharma–Tasso–Olever equation. This method is reliable and effective, and gives several new solution functions such as rational function solutions, single king solution and periodic solutions. We think that the proposed method can also be applied to other generalized fractional nonlinear differential equations. In our future studies, we will solve nonlinear fractional partial differential equations by this approach. It is interesting to point out that the fractional derivative parameter α plays an important role in modulating the amplitude of the soliton solution.

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Hasan Bulut is currently an assistant professor of mathematics in Firat University. He has published more than 30 articles journals. His research interests include stochastic differential equations, fluid and heat mechanics, finite element method, analytical methods for nonlinear differential equations, mathematical physics, and numerical solutions of the partial differential equations, computer programming.

Y. Pandir is an assistant professor in Department of Mathematics at Bozok University; Yozgat (Turkey). He obtained his M.Sc. degree from Celal Bayar University and Ph.D. degree from Erciyes University. His research interests include fluid mechanics, finite element method, analytical methods for nonlinear differential equations, mathematical physics, and numerical analysis. His research papers are published in Abstract and Applied Analysis, Applied Mathematical Sciences, International Journal of Physical Sciences, Physica Scripta and Journal of Mathematics.