

Hopf Bifurcation and Stability Analysis for A Delayed Logistic Equation

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Abstract—This paper deals with the stability analysis and the Hopf bifurcation at the equilibrium points of a logistic delay differential equation. By applying the Halanay inequality, the local stability of the logistic differential equation is discussed. The stability of bifurcation periodic solutions and the direction of Hopf bifurcation are determined by applying the normal form theory and the center manifold theorem. Numerical examples show interesting nonlinear behavior of the logistic differential equation at the end of the paper.

Index Terms—Differential equations with delay, Hopf bifurcation, local stability, global asymptotic stability, periodic solutions.

I. INTRODUCTION

Researches about the delayed logistic equations have received significant attention in the recent years [1]-[6], especially from biologists and mathematicians. Hutchinson [1] assumed egg formation to occur τ units of time before hatching and proposed the following delayed logistic equation:

$$\frac{dx}{dt} = r \cdot x(t) \left\{ 1 - \frac{x(t-\tau)}{K} \right\} \quad (1.1)$$

where $r > 0$ is intrinsic growth rate, $K > 0$ is the carrying capacity of population and $\tau > 0$ is time delay. Properties of Eq. (1.1) were studied by various authors [7]-[9]. It was shown in [4] that for parameters r and τ the equilibrium point $x=K$ is locally asymptotically stable.

K. Gopalsamy [2] considered a realistic and general case of Hutchinson's equation

$$\frac{dx(t)}{dt} = r \cdot x(t) \{ 1 - a_1 \cdot x(t) - a_2 x(t-\tau) \} \quad (1.2)$$

where a_1, a_2 and r are positive constants. By constructing a suitable Liapunov function, Gopalsamy proved that if $r\tau e^{r\tau} < 1$, then the solutions of Eq. (1.2) converge to the positive equilibrium $x_* = \frac{1}{a_1 + a_2}$.

C. Sun *et al.* [5] reconsidered Eq. (1.2) and showed that the positive equilibrium point is globally asymptotically stable and devote their attention to the global existence of periodic

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solutions to Eq. (1.2). M. Jiang *et al.* [6] concerned their attention to the logistic differential equation

$$\frac{dx(t)}{dt} = -\alpha x(t) + r \cdot x(t-\tau) \{ 1 - x(t-\tau) \} \quad (1.3)$$

where α is a known positive parameter, r is an unknown parameter and $\tau > 0$ is time delay. They investigated the linear stability of Eq. (1.3) by analyzing the associated characteristic transcendental equation. M. Jiang *et al.* reconsidered Eq. (1.3) to applied the Halanay inequality and discussed the local stability of (1.3). By constructing numerical examples they also detect nonlinear behaviors with a single parameter delay.

In this paper, we extend the delayed logistic differential equation of (1.2) by adding an extra delayed term as follows

$$\frac{dx}{dt} = r \cdot x(t) \{ 1 - \alpha \cdot x(t) - \beta x(t-\tau) \} + \gamma x(t-\tau) \quad (1.4)$$

where the parameters α, β are known positive real numbers, $r, \gamma \in R - \{0\}$ and $\tau > 0$ is time delay. We emphasize here the two unknown parameters r and γ , which are important to determine the locally asymptotically stability of the equilibrium points of Eq. (1.4), the existence of Hopf bifurcation and the direction of the bifurcating periodic solutions. This paper is organized as follows: In Section II, using the lemma in [10] we show that the equilibrium points of Eq. (1.4) are locally asymptotically stable. Further, by constructing a suitable Liapunov function, we get that the solutions of Eq. (1.4) converge to the equilibrium points under specific conditions. In Section III, the formula are presented for determining bifurcation direction and stability of the bifurcating periodic solutions of Eq. (1.4).

II. LOCAL STABILITY AND EXISTENCE

With the transformation $u(t) = x(t)$, we can rewrite Eq. (1.4) as the delay differential equation

$$\frac{du(t)}{dt} = r\tau u(t) \{ 1 - \alpha u(t) - \beta u(t-1) \} + \gamma\tau u(t-1) \quad (2.1)$$

This equation has two equilibrium points, which are $\bar{u}_1 = 0$ and $\bar{u}_2 = \frac{r+\gamma}{r(\alpha+\beta)}$, where $r \neq 0$. After linearization of Eq. (2.1) at the neighborhood of $\bar{u}_1 = 0$, one gets:

$$\frac{du(t)}{dt} = r\tau u(t) + \gamma\tau u(t-1) \quad (2.2)$$

The characteristic equation of Eq. (2.2) is of the form

$$\lambda - r\tau - \gamma\tau e^{-\lambda} = 0 \tag{2.3}$$

On the other hand, after linearization of Eq. (2.1) at the neighborhood of $\bar{u}_2 = \frac{r+\gamma}{r(\alpha+\beta)}$, one obtains:

$$\frac{dy(t)}{dt} = -\tau(\alpha r \bar{u}_2 + \gamma)y(t) - \tau(\beta r \bar{u}_2 - \gamma)y(t-1), \tag{2.4}$$

where $y(t) = u(t) - \bar{u}_2$. The characteristic equation of Eq. (2.4) is of the form

$$\lambda + \tau(\alpha r \bar{u}_2 + \gamma) + \tau(\beta r \bar{u}_2 - \gamma)e^{-\lambda} = 0 \tag{2.5}$$

Clearly, the stability of the equilibriums $\bar{u}_1 = 0$ and $\bar{u}_2 = \frac{r+\gamma}{r(\alpha+\beta)}$ depend on the roots of the characteristic equations (2.3) and (2.5), respectively.

The following theorems are obtained by using the references [10] and [11]. Therefore, it will be omitted to the readers.

Theorem 2.1. Let $\bar{u}_1 = 0$ and $\bar{u}_2 = \frac{r+\gamma}{r(\alpha+\beta)}$ are the equilibrium points of Eq. (2.1), where $r \neq 0$. The following statements are true.

- 1) Let $r < \frac{1}{\tau}$. The equilibrium point $\bar{u}_1 = 0$ of Eq. (2.1) is locally stable if $-\frac{1}{\tau} \sqrt{r^2 \tau^2 + \chi^2} < \gamma < -r$, where $\chi = r\tau \tan \chi$, $0 < \chi < \pi$.
- 2) Let $\gamma > -\frac{1}{\tau}$. The equilibrium point $\bar{u}_2 = \frac{r+\gamma}{r(\alpha+\beta)}$ of Eq. (2.1) is locally stable if

$$-\gamma < r < \frac{\alpha\gamma}{\beta} + \frac{1}{\beta} \sqrt{((2\alpha + \beta)\gamma + \alpha r)^2 (1 + \tan^2 \chi)}$$

where $\chi = -\frac{\tau}{\alpha + \beta} (\gamma(2\alpha + \beta) + \alpha r) \tan \chi$, $0 < \chi < \pi$.

Theorem 2.2. If $\gamma^2 < r^2$, then the equilibrium point $\bar{u}_1 = 0$ of Eq. (2.1) is asymptotically stable for all delay $\tau > 0$.

Theorem 2.3. If $\beta > \alpha$ and $r \in \left(\frac{4\alpha\gamma - \sqrt{\Delta}}{2(\beta - \alpha)}, \frac{4\alpha\gamma + \sqrt{\Delta}}{2(\beta - \alpha)} \right)$, where $\Delta = 4\gamma^2(\alpha + \beta)^2$, then the equilibrium point $\bar{u}_2 = \frac{r+\gamma}{r(\alpha+\beta)}$ of Eq. (2.1), where $r \neq 0$ is asymptotically stable for all delay $\tau > 0$.

Theorem 2.4. Let $\gamma < 0$, $r > 0$. Suppose for $\tau > 0$ that $0 < x(t) < \sqrt{2r+1} \cdot x(t) < x(t-\tau)$. Then the equilibrium point $\bar{u}_1 = 0$ of Eq. (1.4) is globally asymptotically stable.

Theorem 2.5. Let $0 < r < -\frac{(\alpha + \beta) + 2\alpha\gamma}{2(2\alpha + \beta)} < -\gamma$ and

$\gamma < -\frac{\alpha + \beta}{2\alpha}$. Suppose that $0 < x(t) < x(t-\tau)$ for $\tau > 0$. Then the equilibrium point $\bar{u}_2 = \frac{r+\gamma}{r(\alpha+\beta)}$ of Eq. (1.4) is globally asymptotically stable.

Theorem 2.6. Let $\gamma = -\frac{1}{\tau} \sqrt{r^2 \tau^2 + \chi^2}$, where $\chi = r\tau \tan \chi$ and $0 < \chi < \pi$. Then there is a Hopf bifurcation from the equilibrium $\bar{u}_1 = 0$ to a periodic orbit.

It can be similarly proven that when r passes the critical point

$$r_* = \frac{\alpha\gamma}{\beta} + \frac{1}{\beta} \sqrt{((2\alpha + \beta)\gamma + \alpha r_*)^2 (1 + \tan^2 \chi)},$$

where $\chi = -\frac{\tau}{\alpha + \beta} (\gamma(2\alpha + \beta) + \alpha r_*) \tan \chi$, $0 < \chi < \pi$, there is a Hopf bifurcation from the equilibrium $\bar{u}_2 = \frac{\gamma+r}{r(\alpha+\beta)}$ to a periodic orbit.

III. STABILITY AND DIRECTION OF THE BIFURCATING PERIODIC SOLUTIONS

Let $\gamma = \gamma_* + \nu$, where $\nu \in R$. In this case $\nu = 0$ is one Hopfbifurcation value for Eq. (2.1). For $\mu \in C$, let

$$L_\nu \mu = r\tau\mu(0) + (\gamma_* + \nu) \cdot \tau\mu(-1) \tag{3.1}$$

and

$$F(\nu, \mu) = -\alpha r\tau\mu^2(0) - \beta\tau\mu(0)\mu(-1) \tag{3.2}$$

By the Riesz representation theorem, there exists a function $\eta(\theta, \nu)$ of bound variation for $\theta \in [-1, 0]$, such that

$$L_\nu \mu = \int_{-1}^{\theta} d\eta(\theta, \nu)\mu(\theta) \text{ for } \mu \in C \tag{3.3}$$

In fact, we can choose

$$\eta(\theta, \nu) = r\tau\delta(\theta) + (\gamma_* + \nu) \cdot \tau\delta(\theta + 1) \tag{3.4}$$

where $\theta \in [-1, 0]$ and δ is the Dirac delta function.

For $\mu \in C^1([-1, 0], R)$, define

$$A(\nu)\mu = \begin{cases} \frac{d\mu(\theta)}{d\theta}, & -1 \leq \theta < 0 \\ \int_{-1}^0 d\eta(\xi, \nu)\mu(\xi), & \theta = 0, \end{cases} \tag{3.5}$$

and

$$B(\nu)\mu = \begin{cases} 0, & -1 \leq \theta < 0 \\ F(\nu, \mu), & \theta = 0. \end{cases} \tag{3.6}$$

Hence, (2.1) can be rewritten as

$$\dot{u}_t = A(v)u_t + B(v)u_t \quad (3.7)$$

where $u_t(\theta) = u(t + \theta)$ for $\theta \in [-1, 0]$.

For $\psi \in C^1[-1, 0]$, we define the adjoint operator A^* of A as

$$A^*(v)\psi = \begin{cases} -\frac{d\psi(\xi)}{d\xi}, & 0 < \xi \leq 1 \\ \int_{-1}^0 d\eta(t, v)\mu(-t), & \xi = 0 \end{cases} \quad (3.8)$$

and a bilinear inner product as

$$\langle \psi, \mu \rangle = \bar{\psi}(0)\mu(0) - \int_{-1}^0 \int_{s=0}^{\theta} \bar{\psi}(s - \theta) d\eta(\theta, 0)\mu(s) ds \quad (3.9)$$

To determine the Poincare normal form of A , we need to calculate the eigenvector q of A belonging to the eigenvalue $i\omega_0$ and eigenvector q^* of A^* belonging to the eigenvalue $i\omega_0$. It can be verified that $q(\theta) = e^{i\omega_0\theta}$ is an eigenvector of $A(0)$ corresponding to $i\omega_0$, and $q^*(s) = De^{i\omega_0s}$ is an eigenvector of $A^*(0)$ with respect to the eigenvalue $i\omega_0$. Furthermore, by (3.8), we have $A^*q^*(\theta) = -i\omega_0q^*(\theta)$.

We normalize q and q^* by the condition $\langle q^*, q \rangle = 1$ to determine the value of D . From (3.9), we have

$$\begin{aligned} \langle q^*, q \rangle &= \bar{D} - \int_{-1}^0 \int_{s=0}^{\theta} \bar{q}^*(s - \theta) d\eta(\theta, 0)\mu(s) ds \\ &= \bar{D}(1 + \gamma_* \tau e^{-i\omega_0}) = 1. \end{aligned}$$

Hence

$$q^*(\theta) = \frac{1}{1 + \gamma_* \tau e^{i\omega_0}} e^{i\omega_0\theta}, \quad 0 \leq \theta \leq 1. \quad (3.10)$$

Similarly, we have $\langle q^*, \bar{q} \rangle = 0$.

To describe the center manifold C_0 at $v = 0$, we want to compute the coordinates. For each $u \in \text{Dom}(A)$, we associate the pair (z, s) , where $z = \langle q^*, u_t \rangle$ and

$$s(t, \theta) = u_t(\theta) - zq(\theta) - \bar{z}\bar{q}(\theta) = u_t(\theta) - 2\text{Re}\{zq(\theta)\} \quad (3.11)$$

On the center manifold C_0 , we have $s(t, \theta) = s(z(t), \bar{z}(t), \theta)$, where

$$s(z, \bar{z}, \theta) = s_{20}(\theta)\frac{z^2}{2} + s_{11}(\theta)z\bar{z} + s_{02}(\theta)\frac{\bar{z}^2}{2} + s_{30}(\theta)\frac{z^3}{6} + \dots \quad (3.12)$$

Here are z and \bar{z} local coordinates for center manifold C_0 in the direction of q and q^* . Note that $s(t, \theta)$ is real, if u_t is

real. We only deal with real solutions. It is easy to show that $\langle q^*, s \rangle = 0$. For solutions $u_t \in C_0$ of Eq. (2.1) we have

$$\langle q^*, \dot{u}_t \rangle = \langle q^*, Au_t + Bu_t \rangle.$$

Since $v = 0$, we have

$$\begin{aligned} \dot{z}_t &= \langle q^*, \dot{u}_t \rangle = \langle q^*, Au_t + Bu_t \rangle \\ &= i\omega_0 z_t + \bar{q}^*(0)f(z, \bar{z}). \end{aligned}$$

We rewrite the above differential equation as

$$\dot{z}(t) = i\omega_0 z + \bar{q}^*(0)f(z, \bar{z}) = i\omega_0 z + g(z, \bar{z}) \quad (3.13)$$

where

$$g(z, \bar{z}) = \bar{q}^*(0)f(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{30}\frac{z^3}{6} + \dots \quad (3.14)$$

From (3.11), we have

$$\begin{aligned} u_t(\theta) &= s(t, \theta) + 2\text{Re}\{zq(\theta)\} = s_{20}(\theta)\frac{z^2}{2} + \\ &+ s_{11}(\theta)z\bar{z} + s_{02}(\theta)\frac{\bar{z}^2}{2} + e^{i\omega_0\theta}z + e^{-i\omega_0\theta}\bar{z} + \dots \end{aligned} \quad (3.15)$$

Using (3.2) and (3.15) in (3.14), we get

$$\begin{aligned} g(z, \bar{z}) &= \bar{q}^*(0)f(z, \bar{z}) \\ &= \bar{q}^*(0)F(0, u_t) \\ &= -\bar{D}r\tau[\alpha u_t^2(0) + \beta u_t(0)u_t(-1)] \end{aligned} \quad (3.16)$$

Comparing the coefficients with (3.14) and (3.16), we have

$$\begin{aligned} g_{20} &= -2\bar{D}r\tau(\alpha + \beta e^{-i\omega_0}) \\ g_{11} &= -2\bar{D}r\tau(\alpha + \beta \cos \omega_0) \\ g_{02} &= -2\bar{D}r\tau(\alpha + \beta e^{i\omega_0}) \\ g_{21} &= -\bar{D}r\tau\left[2\alpha(s_{20}(0) + 2s_{11}(0)) + \right. \\ &\left. \beta(e^{i\omega_0}s_{20}(0) + s_{20}(-1) + 2s_{11}(-1) + 2e^{-i\omega_0}s_{11}(0))\right] \end{aligned}$$

Since s_{20} and s_{11} are unknown in g_{21} , we want to compute and determine them. From (3.11) and (3.13), we obtain

$$\begin{aligned} \dot{s} &= \dot{u}_t - q\dot{z} - \bar{q}\dot{\bar{z}} \\ &= Aw - 2\text{Re}\{\bar{q}^*f(z, \bar{z})q\} + Bu_t. \end{aligned}$$

So, we have by the definition of B in Eq. (3.6),

$$\dot{s} = \dot{u}_t - q\dot{z} - \bar{q}\dot{\bar{z}} = \begin{cases} As - 2\text{Re}\{\bar{q}^*f(z, \bar{z})q(\theta)\}, & -1 \leq \theta < 0 \\ As - 2\text{Re}\{\bar{q}^*f(z, \bar{z})q\} + f(z, \bar{z}) & \theta = 0. \end{cases} \quad (3.17)$$

We rewrite (3.17) as

$$\dot{s}(t) = As + K(z, \bar{z}, \theta) \tag{3.18}$$

where

$$K(z, \bar{z}, \theta) = K_{20}(\theta) \frac{z^2}{2} + K_{11}(\theta) z\bar{z} + K_{02}(\theta) \frac{\bar{z}^2}{2} + \dots \tag{3.19}$$

Substituting the corresponding series into (3.17) and comparing the coefficients, we obtain

$$(A - 2i\omega_0)s_{20}(\theta) = -K_{20}(\theta), \quad As_{11}(\theta) = -K_{11}(\theta), \dots \tag{3.20}$$

Using (3.19) in (3.17), we have

$$K_{20}(\theta) \frac{z^2}{2} + K_{11}(\theta) z\bar{z} + K_{02}(\theta) \frac{\bar{z}^2}{2} + \bar{q}^*(0)f(z, \bar{z})q + q^*(0)\overline{f(z, \bar{z})q} + \dots = 0,$$

which can also be written as

$$K_{20}(\theta) \frac{z^2}{2} + K_{11}(\theta) z\bar{z} + K_{02}(\theta) \frac{\bar{z}^2}{2} + \dots + g(z, \bar{z})q + \overline{g(z, \bar{z})q} = 0.$$

Comparing these coefficients with those of (3.14), we get

$$\begin{cases} K_{20}(\theta) = -g_{20}q(\theta) - \overline{g_{02}} \cdot \bar{q}(\theta), \\ K_{11}(\theta) = -g_{11}q(\theta) - \overline{g_{11}}\bar{q}(\theta), \\ K_{02}(\theta) = -g_{02}q(\theta) - \overline{g_{20}}\bar{q}(\theta), \\ \dots \end{cases} \tag{3.21}$$

From (3.17), (3.21) and by the definition of A, that we have

$$\dot{s}_{20}(\theta) = 2i\omega_0 s_{20} + g_{20}q(\theta) + \overline{g_{02}}\bar{q}(\theta) \tag{3.22}$$

and

$$\dot{s}_{11}(\theta) = g_{11}q(\theta) + \overline{g_{11}}\bar{q}(\theta) \tag{3.23}$$

Since $q(\theta) = e^{i\omega_0\theta}$, solving (3.22) and (3.23), we obtain

$$s_{20}(\theta) = i \frac{g_{20}}{\omega_0} e^{i\omega_0\theta} + i \frac{\overline{g_{02}}}{3\omega_0} e^{-i\omega_0\theta} + U_1 e^{2i\omega_0\theta} \tag{3.24}$$

and

$$s_{11}(\theta) = -i \frac{g_{11}}{\omega_0} e^{i\omega_0\theta} + i \frac{\overline{g_{11}}}{\omega_0} e^{-i\omega_0\theta} + U_2 \tag{3.25}$$

where U_1 and U_2 are constants. From the definition of A, (3.17) and (3.20), we can write

$$\begin{cases} K_{20}(0) = -g_{20} - \overline{g_{02}} - 2r\tau(\alpha + \beta e^{-i\omega_0}) = -As_{20}(0) + 2i\omega_0 s_{20}(0), \\ K_{11}(0) = -g_{11} - \overline{g_{11}} - 2r\tau(\alpha + \beta \cos \omega_0) = -As_{11}(0), \\ \dots \end{cases} \tag{3.26}$$

From the first equation of system (3.26), we obtain

$$-g_{20} - \overline{g_{02}} - 2r\tau(\alpha + \beta e^{-i\omega_0}) = -r\tau s_{20}(0) - \gamma_* \tau s_{20}(-1) + 2i\omega_0 s_{20}(0)$$

From this, we get

$$U_1 = \frac{ig_{20} \left(\frac{r\tau + \gamma_* \tau e^{-i\omega_0}}{\omega_0} - i \right) + i\overline{g_{02}} \left(\frac{r\tau + \gamma_* \tau e^{i\omega_0}}{3\omega_0} + \frac{i}{3} \right) - 2r\tau(\alpha + \beta e^{-i\omega_0})}{2i\omega_0 - r\tau - \gamma_* \tau e^{-2i\omega_0}} \tag{3.27}$$

By the second equation of system (3.26), we have

$$-g_{11} - \overline{g_{11}} - (\alpha + \beta \cos \omega_0) = -r\tau s_{11}(0) - \gamma_* \tau s_{11}(-1).$$

Computing it, we obtain

$$U_2 = \frac{ig_{11} \left(\frac{r\tau + \gamma_* \tau e^{-i\omega_0}}{\omega_0} - i \right) - i\overline{g_{11}} \left(\frac{r\tau + \gamma_* \tau e^{i\omega_0}}{\omega_0} + i \right) + 2r\tau(\alpha + \beta \cos \omega_0)}{\tau(r + \gamma_*)} \tag{3.28}$$

Finally, from (3.24), (3.25), (3.27) and (3.28), we can see that each g_{ij} ($i, j = 0, 1, 2, \dots$) is determined by the parameters and delays in Eq. (2.1). Thus, we can compute the following values

$$\begin{cases} c_1(0) = \frac{i}{2\omega_0} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2}, \\ \mu_2 = -\frac{\text{Re}(c_1(0))}{\text{Re}(\lambda'(\gamma_*))}, \\ \beta_2 = 2 \text{Re}(c_1(0)). \end{cases} \tag{3.28}$$

A general result for the direction and stability of Hopf bifurcation ([12], Chapter 1, Section IV), implies that the direction of the Hopf bifurcation is determined by the sign of μ_2 and the stability of bifurcating periodic solutions by the sign of β_2 , respectively. In this case, if $\mu_2 > 0$ (< 0), then the Hopf bifurcation is supercritical (subcritical) and if $\beta_2 < 0$ (> 0) the bifurcating periodic solution are orbitally stable (unstable).

IV. EXAMPLE

1) From Theorem 2.6, using $\tau = 0.5$, $r = -4.71238898025$, $\alpha = 0.5$, $\beta = 0.8999999$, $\gamma = -6.66432399526318$ and $x(0) = 0.8001$, the graph of the first 50 iterations of Eq. (1.4) is given in Fig. 1 (a) and Fig. 1 (b). These figures show us that there is a Hopf bifurcation from the equilibrium point $\bar{u}_1 = 0$ of Eq. (2.1) to a periodic orbit.

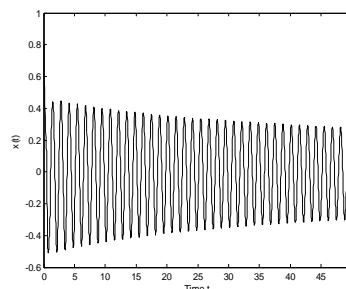


Fig. 1 (a)

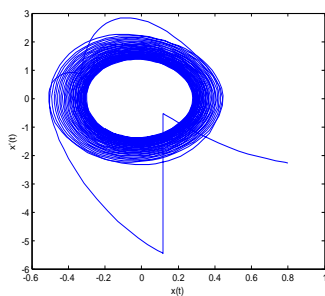


Fig. 1 (b)

2) Using $\tau = 0.5$, $r = 1.5707963275$, $\alpha = 0.0005$, $\beta = 0.0003$, $\gamma = -2.221441470076$ and $x(0) = 0.8001$, the graph of the first 50 iterations of Eq. (1.4) is given in Fig. 2 (a) and Fig. 2 (b). We obtain that there is a Hopf bifurcation from the equilibrium point \bar{u}_2 of Eq. (2.1) to a periodic orbit.

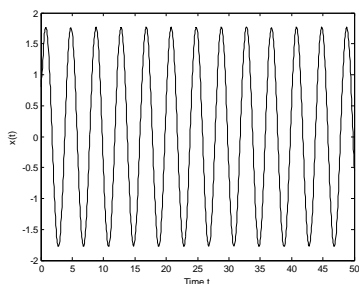


Fig. 2 (a)

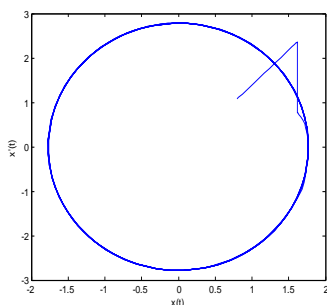


Fig. 2(b)

V. DISCUSSION

It can be shown that if we use the parameters in Example 1), to get the values of (3.28), then we obtain $C_1(0) = -0.64676 - 0.49156i$, $\mu_2 = -5.3852$ and $\beta_2 = -1.29352$. In this case, Fig. 1 (a) and Fig. 1 (b) show us that the Hopf bifurcation is subcritical and that bifurcating periodic solution are orbitally stable.

Similarly, in Example 2), computations give that $C_1(0) = 0.00018155 - 0.00055611i$, $\mu_2 = 0.00059634$ and

$\beta_2 = 0.0003631$, which show that in Fig. 2(a) and Fig. 2 (b), we have a supercritical Hopf bifurcation. Therefore, these bifurcating periodic solutions are orbitally unstable.

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