Analysis of Delayed Toxin Producing Phytoplankton-Zooplankton System

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Abstract—**A dynamical model for toxin producing phytoplankton and zooplankton has been formulated and analyzed. Due to gestation of prey, a discrete time delay is incorporated in the predator dynamics. The stability of the delay model is discussed and Hopf-bifurcation to a periodic orbit is established. To substantiate analytical findings, numerical simulations are performed. The system shows rich dynamic behavior including chaos and limit cycles.**

Index Terms—**Chaos, hopf bifurcation, periodic solution, time delay.**

I. INTRODUCTION

Phytoplanktons are tiny floating plants (algae) that live near the surface of lakes and ocean. They are at the lowest trophic level of aquatic ecosystem [1]. In the process of photosynthesis, phytoplankton produces half of the world's oxygen. The nature of rapid increase and decrease in phytoplankton population density is the main characteristics in plankton ecosystem. Although the sudden appearance and disappearance of blooms is not well understood but the adverse effects of harmful algal blooms (HAB) on human health, aquatic population, fisheries business, and tourism are well established [2].

The dynamical interaction of zooplankton and phytoplankton on the occurrence of bloom is of interests to many scientific investigations. Toxins are produced by phytoplankton to avoid predation by zooplankton. The toxin producing phytoplankton not only reduces the grazing pressure on them but can also control the occurrence of bloom, Sarkar and Chattopadhyay [3]-[4].
The ubiquity of time-delay-coupled systems is suggestive of its importance, applicability, and utility in a large range of biological systems. Models incorporating time delays in diverse biological models are extensively reviewed by Beretta and Kuang [5], Gopalsammy [6], and Cushing [7]. The discrete time delay has potential to change the qualitative behavior of the dynamical systems [8]-[9].

In this present paper a delay differential model is proposed to study the interaction of a zooplankton and toxin producing phytoplankton. The discrete time delay is considered in predators' functional response due to gestation time.

II. BASIC MATHEMATICAL MODEL

Consider a toxin producing phytoplankton P and zooplankton Z in marine ecosystem. The growth of phytoplankton is assumed to be logistic in the absence of predation. The predation rate is assumed to follow the law of mass action. The effect of toxin liberation decreases the growth of zooplankton according to Holling type II functional response. Accordingly, the following basic model for the dynamics of phytoplankton- zooplankton system in the absence of delay and seasonality is given as:

$$\frac{dP}{dt} = rP\left(1-\frac{P}{K}\right) - \alpha PZ$$
$$\frac{dZ}{dt} = \beta PZ - Z\mu - \frac{\theta PZ}{(P+\gamma)} \tag{1}$$

subject to initial conditions $P(0) \geq 0$ and $Z(0) \geq 0$.

Here r represents the intrinsic growth rate of phytoplankton and K is the carrying capacity of phytoplankton in absence of predation. The constant μ is the mortality rate of zooplankton, $\alpha > 0$ is the specific predation rate and $\beta > 0$ represents the ratio of biomass consumed per-zooplankton for the self-production. The last term in the zooplankton dynamics represents the loss due to toxin production by phytoplankton. The constant $\gamma > 0$ is the half saturation level and $\theta > 0$ is the rate of release of toxicants per phytoplankton population.

The local dynamics of autonomous system (1) is summarized below in the form of following results which will be used later are stated here:

1) All solutions of the system (1) are uniformly bounded within a domain

$$B = \left\{ (P,Z) \in R_+^2 : \beta P + \alpha Z \leq \frac{\beta K(\mu+r)^2}{4r\mu} \right\} \subseteq R^2$$

2) The boundary equilibrium point $E_0(0,0)$ has a locally unstable manifold along P axis and stable manifold along Z axis. Therefore, for small population sizes, the phytoplankton is increasing along P axis and the zooplankton is decreasing along Z axis.

3) Another boundary equilibrium point $E_1 = (K,0)$ is stable when

$$\mu > K\left(\beta - \frac{\theta}{(\gamma+K)} \right)$$

4) The interior equilibrium point $E^* = (P^*, Z^*)$;

$$P^* = \frac{1}{2\beta}\left((\mu+\theta-\beta\gamma)+\left\{(\mu+\theta-\beta\gamma)^2+4\beta\gamma\mu\right\}^{1/2}\right)$$

$$Z^* = \frac{r}{\alpha}\left(1-P^*/K\right)$$

exists provided $P^* < K$ and $\beta < \dfrac{\mu+\theta}{\gamma}$ (2)

5) It is observed that E_1 is locally asymptotically stable whenever E^* does not exist and a saddle point otherwise.

6) The positive equilibrium point E^* is locally asymptotically stable provided

$$\beta > \frac{\theta\gamma}{(P^*+\gamma)^2} \tag{3}$$

7) Further, the system (1) around E^* is globally asymptotically stable [5].

For explanation of occurrence of bloom phenomenon in phytoplankton-zooplankton system, the time-lag and periodic external forces are incorporated in the system (1).

III. MODEL WITH DISCRETE DELAY

Consider the discrete time delay τ in zooplankton dynamics due to gestation of phytoplankton, accordingly the system (1) is extended in delay differential equation system as:

$$\frac{dP}{dt} = rP\left(1-P/K\right)-\alpha PZ$$

$$\frac{dZ}{dt} = \beta P(t-\tau)Z(t-\tau)-Z\mu-\frac{\theta PZ}{(P+\gamma)} \tag{4}$$

The initial conditions for system (4) are:

$$P(\phi)\geq 0,\ Z(\phi)\geq 0;\ \phi\in[-\tau,0)$$

The system (1) and (4) has same equilibrium points. The stability of boundary equilibrium points are not affected due to delay. For stability of interior equilibrium point $E^*=(P^*,Z^*)$, introduce the perturbations as

$$P(t) = P^* + p(t),\ Z(t) = Z^* + z(t)$$

Then the system (4) gets transformation to

$$\frac{dp}{dt} = ap(t)+bz(t)$$

$$\frac{dz}{dt} = cp(t)+dp(t-\tau)+ez(t)+fz(t-\tau)$$

$$a = -r\frac{P^*}{K},\ b = -\alpha P^*,\ c = -\frac{\theta\gamma Z^*}{(P^*+\gamma)^2},\ d = \beta Z^*,$$

$$e = -\mu-\frac{\theta P^*}{(P^*+\gamma)},\ f = \beta P^*$$

The characteristic equation corresponding to (4) is given By

$$G(\lambda,\tau)\equiv\lambda^2-\lambda(a+e)+(ae-bc)-e^{-\lambda\tau}[(bd-af)+\lambda f]=0 \tag{6}$$

IV. LOCAL HOPF BIFURCATION

It is observed that the positive equilibrium point $E^*=(P^*,Z^*)$ of the system (4) with $\tau=0$ is stable for the parameters satisfying conditions (3). However, for $\tau>0$, E^* will be locally asymptotically stable if the real part of $G(\lambda,\tau)$ is negative.

For Hopf bifurcation, it is assumed that root of characteristic equation (6) is purely imaginary. Substituting in (6) and separating real and imaginary parts, the following is obtained:

$$-\omega^2+(ae-bc) = (bd-af)\cos\omega\tau+\omega f\sin\omega\tau$$

$$-\omega(a+e) = \omega f\cos\omega\tau-(bd-af)\sin\omega\tau$$

Simplification thereafter, leads to the following cubic equation

$$\upsilon^2+Q_1\upsilon+Q_2 = 0 \tag{7}$$

With $\upsilon=\omega^2$ and

$$Q_1 = [(a+e)^2-f^2-2(ae-bc)]>0$$

$$Q_2 = (ae-bc)^2-(bd-af)^2$$

The equation (7) has a unique positive root when $Q_2<0$. The critical value of delay is computed as

$$\tau_k = \frac{1}{\omega_0}\cos^{-1}\left[\frac{-\omega_0^2(bd+ef)+(ae-bc)(bd-af)}{(bd-af)^2+\omega_0^2 f^2}\right]+\frac{2k\pi}{\omega_0} \tag{8}$$
$$k = 0,1,2,...$$

$$\omega_0 = \frac{\sqrt{-Q_1+\sqrt{Q_1^2-4Q_2}}}{2} \tag{9}$$

Suppose $\lambda_k(\tau)=u_k(\tau)+\iota\omega_k(\tau_k)$ be a root of (6) near $\tau=\tau_k$, then substituting $\lambda_k(\tau)=u_k(\tau)+\iota\omega_k(\tau_k)$ in the characteristic equation and separating real and imaginary parts

$$H_1(u_k,\omega_k,\tau) = u_k^2+\omega_k^2-u_k(a+e)+(ae-bc)-e^{-u_k\tau}(bd-af)$$
$$\cos\omega_k\tau-u_k fe^{-u_k\tau}\cos\omega_k\tau-\omega_k f\sin\omega_k\tau$$

$$H_2(u_k,\omega_k,\tau) = 2u_k\omega_k-\omega_k(ae-bc)+e^{-u_k\tau}(bd-af)\sin\omega_k\tau+$$
$$u_k fe^{-u_k\tau}\sin\omega_k\tau-\omega_k f\cos\omega_k\tau-\omega_k fe^{-u_k\tau}$$

Observe that $H_1(0,\omega_k,\tau_k)=H_2(0,\omega_k,\tau_k)=0$ and

$$J = \begin{pmatrix} \dfrac{\partial H_1}{\partial u_k} & \dfrac{\partial H_1}{\partial\omega_k} \\ \dfrac{\partial H_2}{\partial u_k} & \dfrac{\partial H_1}{\partial\omega_k} \end{pmatrix}\neq 0$$

By implicit function theorem, u_k, ω_k is a function of τ in a neighborhood of $(0, \omega_k, \tau_k)$ such that $u_k(\tau_k) = 0$, $\omega_k(\tau_k) = \omega_k$, $k = 0,1,2,...$

Differentiation of characteristic equation (6) with respect to τ gives

$$[2\lambda - (a+e)]\frac{d\lambda}{dt} = [(bd-af) + \lambda f(-\tau e^{-\lambda\tau}) + e^{-\lambda\tau}f]\frac{d\lambda}{dt}$$
$$- \lambda e^{-\lambda\tau}[(bd-af) + \lambda f]$$

Or

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(bd-af)}{-\lambda^2[(bd-af)+\lambda f]} -$$
$$\frac{\lambda^2-(ae-bc)}{\lambda^2[\lambda^2-\lambda(a+e)+(ae-bc)]} - \frac{\tau}{\lambda}$$

By Rouche's Theorem [10]

$$sign\left\{\frac{d(\text{Re}\,\lambda)}{d\tau}\right\}_{\lambda=i\omega_k} = sign\left\{\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_k}$$

Thus

$$\left[\frac{d(\text{Re}\,\lambda(\tau))}{d\tau}\right]^{-1} = sign\left\{\text{Re}\left[\frac{(bd-af)}{-\lambda^2[(bd-af)+\lambda f]}\right]_{\lambda=i\omega_k}\right.$$

$$\left.-\text{Re}\left[-\frac{\lambda^2-(ae-bc)}{\lambda^2[\lambda^2-\lambda(a+e)+(ae-bc)]}\right]_{\lambda=i\omega_k}\right\}$$

$$= sign\left[\frac{\omega_k^4-(ae-bc)^2}{\omega_k^2[(\omega_k^2-(ae-bc))^2+\omega_k^2(a+e)^2]}\right.$$

$$\left.+\frac{(bd-af)^2}{\omega_k^2(bd-af)^2+\omega_k^2 f^2}\right]$$

$$= sign\left[\frac{\omega_k^4+(bd-af)^2-(ae-bc)^2}{\omega_k^2[\omega_k^2(bd-af)^2+\omega_k^2 f^2]}\right]$$

Since $Q_2 < 0$ i.e. $(bd-af)^2 - (ae-bc)^2$ implies

$$\left.\frac{d(\text{Re}\,\lambda)}{d\tau}\right|_{\lambda=i\omega_k,\,\tau=\tau_k} > 0.$$

Thus, the transversality condition holds and the conditions for Hopf bifurcation are satisfied at $\tau = \tau_k, k = 0,1,2,...$

The following theorem on stability and Hopf bifurcation of system (4) can now be stated:

Theorem: Let the positive equilibrium point $E^* = (P^*, Z^*)$ of system (4) with $\tau = 0$ is stable then,

(i) If $Q_2 > 0$, E^* is locally asymptotically stable for all $\tau \geq 0$.

(ii) If $Q_2 < 0$, then E^* is locally asymptotically stable for all $\tau \in [0, \tau_k)$ and unstable whenever $\tau > \tau_k$ for $k = 0,1,2,...$

(iii) System undergoes Hopf-bifurcation around E^* for every $\tau = \tau_k$ for $Q_2 < 0$.

V. NUMERICAL COMPUTATION

The numerical simulations have been carried out to substantiate our analytical findings and investigate the global dynamical behavior of the nonlinear coupled system (4). The possibility of existence of chaotic or the quasi periodic solution of the system has been explored. Consider the following set of parameters values [3]-[4]:

$$r = 1.5h^{-1},\ K = 400l^{-1},\ \alpha = 0.068lh^{-1},\ \beta = 0.08lh^{-1},$$
$$\mu = 1.5h^{-1},\ \gamma = 0.5l^{-1},\ \theta = 4h^{-1}$$
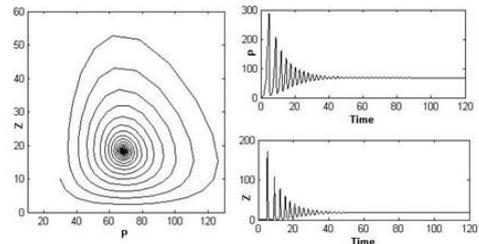
Fig. 1. Phase plot and time series depicting stable behavior at $\tau = 0.01h$

For this choice of parameters, the analysis suggests that the Hopf-bifurcation occurs at critical value of parameter, computed from (8), as $\tau \approx 0.056h$.
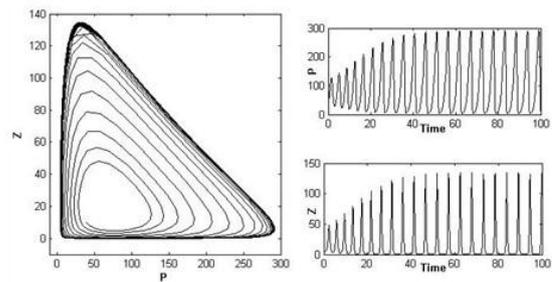
Fig. 2. Phase plot and time series depicting limit cycle at $\tau = 0.125h$

The system (4) is solved numerically for the above choice of parameters with $\tau = 0.125h > \tau_0$. The phase plot and the two time series are drawn in Fig.1. The solution trajectory approaches to the equilibrium state $(68.39, 18.29)$ asymptotically confirming the coexistence of species for this choice of parameters.

It is observed in the simulation experiment that the system loses its stability and limit cycle is obtained as the delay is increased, keeping all other parameters fixed. The phase plot and the time series are drawn for $\tau = 0.125h > \tau_0$ in Fig. 2. The solution trajectory tends to a stable limit cycle in this case.
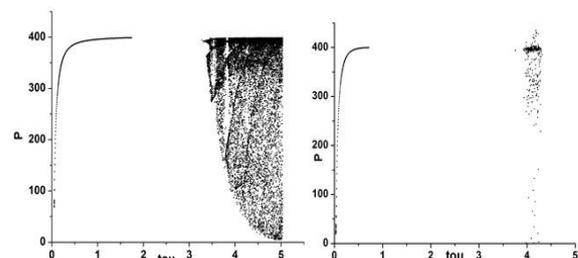
Fig. 3. Bifurcation diagram w. r. t. τ at (a) $\theta = 4.0h^{-1}$ (b) $\theta = 4.0$

To obtain bifurcation diagram, the local maximum values of a species is plotted for a range of the key parameter for fixed initial conditions after removing the transient effects in the solution. The bifurcation diagram is drawn with respect to τ in Fig. 3 (a). A small window of stability is observed near $\tau = 0.056 h$ and then limit cycle is obtained. This is in confirmation with the analytical results. Another window of stability is clearly visible after which the solution undergoes series of period doubling leading to chaos.

The bifurcation diagram is also plotted for τ at $\theta = 0$ in Fig. 3 (b). Comparison with Fig. 3(a) clearly shows that the toxin production enhances the complexity in the system. It is observed that stable window narrows down with increasing θ and chaotic region widens up.

In Fig. 4 (a), phase diagram shows a typical chaotic attractor for $\theta = 4h^{-1}$ at $\tau = 4 h$. In the same way, another attractor is obtained for $\theta = 0$. It is evident that the toxin production increases the complexity in the zooplankton phytoplankton system enhancing the possibility of algal blooms.

The next bifurcation diagram is plotted for the toxin parameter θ at $\tau = 3.5 h$ in Fig. 5. This Bifurcation diagram clearly shows the region of choas for value of $\tau = 3.5 h$.

In this diagram, several stable windows appeared for $\theta < 2.4$ and the prominent ones are in the interval $(0, 0.35)$ and $(2.37, 3.33)$. A periodic window is observed in the interval $(3.3, 4.5)$. After periodic doubling solution becomes chaotic with further increase in θ.

VI. CONCLUSIONS

In this paper, a delay model for toxin producing phytoplankton and zooplankton has been investigated. The discrete time delay is incorporated due to gestation delay. Local asymptotic stability condition for nontrivial equilibrium point in the absence of delay is established. Hopf bifurcation analysis gives the critical time delay for which periodic solutions are possible. Thus, the species can coexist along with toxin liberation. Further increasing the delay beyond the critical value, the system goes periodic. The chaotic behavior is observed in the system which indicates the occurrence of algal blooms.
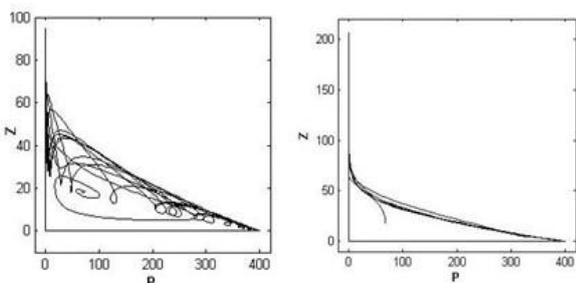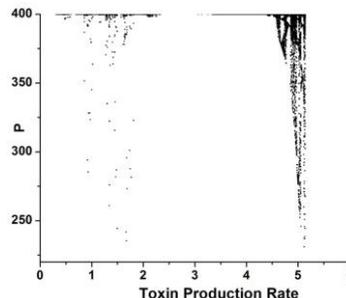
Fig. 5. Bifurcation diagram w. r. t. θ for $\tau = 3.5 h$

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Fig. 4. Attractor at $\tau = 4 h$ for (a) $\theta = 4 h^{-1}$ (b) $\theta = 0$