Practical Expressions of Elastoplastic Contact between Rough Surfaces

Hongliang Tian, Chunhua Zhao, Dalin Zhu, and Hongling Qin

***Abstract*—A difficulty associated with widespread acceptance GW model is that there is no analytic expression for the parameters of interest at the interface. The probability density of asperity heights for nearly 90% of engineering surfaces tends to be Gaussian. Applying GW model simple exponential probability density to approximate the Gaussian one is not explicit for some cases. The evaluation of parabolic cylinder function was carried out by employing software Maple. Some practical closed form analytic expressions were derived for the contact load, contact area and contact spot number for both GW elastic contact model and CEB elastoplastic contact model by the generalized exponential probability density fitting the Gaussian one. Some main digital features of the two contacting rough surfaces were given in the tabulating form.**

***Index Terms*—Surface roughness, partial lubrication regime, surface separation, individual asperity**

I. Introduction

It is well known that most engineering surfaces are rough. Although the contact behavior between two such surfaces is complex, understanding the nature of the contact phenomena is pivotal to gaining insights into interfacial behavior, such as thermal resistance, electrical resistance, and wear. The milestone was set forth in the classic paper of Greenwood and Williamson who introduced a basic elastic dry contact GW model [1]. McCOOL [2] numerically compared the basic GW elastic microcontact model with two more general isotropic and anisotropic models and suggested after a series of detailed numerical examples that GW model, in despite of its simplistic form, gave good order-of-magnitude estimates of the number of contacts, real contact area fraction and nominal pressure, thus justifying its practical use. CEB model [3] extended GW model to involve elastoplastic deformations of the asperities. Lee and Ren [4] gave numerical solutions of elastoplastic rough surfaces. Following the original GW model, numerous researchers used the simple exponential probability density of asperity heights in order to attain closed form expressions. For example, Hess and Soom [5], [6] and Hess and Wagh [7] applied the simple exponential probability density to dynamic friction model with angular motions. Etsion and Front employed the same density and made a reasonable assumption in static sealing flow of end face seals [8]. A different method had been adopted by Bhushan [9] who empirically used least square method to fit the results of GW model instead of statistical distribution of asperity heights to numerically calculate. 15 years later, of particular interest is the pioneering work of Polycarpou and Etsion [10] who theoretically checked two fitting constants of Ref. [9] by applying the modified exponential function. An original work of Lin and Lin [11] derived the maximum contact pressure factor different from CEB model by the method of curve fitting. Antoine et al. [12] obtained numerical approximate solution for Hertzian contact theory by using approximates of elliptical functions. Farhang and Lim [13] derived approximate closed-form equations for normal and tangential contact forces of rough surfaces in dry friction. Akbarzadeh and Khonsari [14] simplified parabolic cylinder function by using the polynomial. Jeng and Peng [15] investigated the microcontact behavior of rough surfaces consisting of elliptical asperities with non-Gaussian height distributions. Zhang and Zhao [16] gave a general distribution of microasperity heights by introducing the roughness exponent α where we can obtain the Gaussian and simple exponential distribution function when $\alpha = 1/2$ and $\alpha = 1$, respectively. Zhao et al. [17] presented an exponential distribution of asperity peak height different from simple exponential one. Zhao et al. [18] constructed a function to satisfy four boundary contact conditions by mapping an appropriate "template" cubic polynomial segment into the quadrilateral bounding.

Based on experimental evidence of cumulative height distribution of mild steel specimen, GW model showed that the height distribution of the asperities for nearly 90% of engineering surfaces tends to be Gaussian. Although in the cases where the asperity heights are at first sight highly non-Gaussian, the uppermost peaks would behave in contact as if Gaussian. A Gaussian distribution would require numerical evaluations and does not allow closed form solution of the contact equations. GW model used a Monte Carlo method to calculate the parabolic cylinder function while the method does not do with the small probability problem [19]. In the manuscript, parabolic cylinder function was evaluated by using software Maple. Some practical closed form analytic solutions were derived for the contact load, contact area and contact spot number for both GW elastic contact model and CEB elastoplastic contact model by the generalized exponential probability density fitting the Gaussian one. Some main digital features of the two contacting rough surfaces were given in the tabulating form.

II. General Equations of Two Contact Surfaces

A. Elastic contact of two Rough Surfaces

The contact spot number, elastic contact area and elastic contact load of the two contacting rough surfaces are given below, respectively

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The authors are with College of Mechanical and Material Engineering, China Three Gorges University, 443002, Yichang city Hubei province, the People's Republic of China (e-mail: thl19732003@yahoo.com.cn).

$$n = \eta A_{\mathrm{n}} F_0(d^*) \tag{1}$$

$$A_{\mathrm{e}} = \pi\beta A_{\mathrm{n}} F_1(d^*) \tag{2}$$

$$P_{\mathrm{e}} = \frac{4}{3}\beta E'\sqrt{\sigma/R}A_{\mathrm{n}} F_{1.5}(d^*) \tag{3}$$

where η denotes the areal density of asperities, A_{n} the nominal area, $\beta = \eta R\sigma$ the roughness parameter, R the radius of curvature of asperity summits, σ standard deviation of asperity heights and the integral is given by

$$F_i(d^*) = \int_{d^*}^{+\infty} (z^*-d^*)^i \phi^*(z^*)\,\mathrm{d}z^* \tag{4}$$

B. Elastic-Plastic Contact of two Rough Surfaces

The dimensionless contact spot number, elastic contact area, plastic contact area, elastic contact load and plastic contact load of the two contacting rough surfaces are given below, respectively

$$n^* = F_0(d^*) \tag{5}$$

$$A_{\mathrm{e}}^* = \pi\beta\int_{d^*}^{d^*+w_{\mathrm{c}}^*} (z^*-d^*)\phi^*(z^*)\,\mathrm{d}z^* \tag{6}$$

$$A_{\mathrm{p}}^* = \pi\beta\int_{d^*+w_{\mathrm{c}}^*}^{+\infty} [2(z^*-d^*)-w_{\mathrm{c}}^*]\phi^*(z^*)\,\mathrm{d}z^* \tag{7}$$

$$P_{\mathrm{e}}^* = \frac{4}{3}\beta\sqrt{\sigma/R}\int_{d^*}^{d^*+w_{\mathrm{c}}^*} (z^*-d^*)^{1.5}\phi^*(z^*)\,\mathrm{d}z^* \tag{8}$$

$$P_{\mathrm{p}}^* = \pi\beta K\frac{H}{E'}\int_{d^*+w_{\mathrm{c}}^*}^{+\infty} [2(z^*-d^*)-w_{\mathrm{c}}^*]\phi^*(z^*)\,\mathrm{d}z^* \tag{9}$$

where H is the hardness of the softer material and d^* the dimensionless surface separation based on asperity heights. The dimensionless critical interference at the inception of plastic deformation and maximum contact pressure factor are provided below, respectively

$$w_{\mathrm{c}}^* = \frac{\pi^2K^2H^2}{4E'^2}\frac{R}{\sigma} = \frac{1}{\psi^2} \tag{10}$$

$$K = 0.454 + 0.41\mu \tag{11}$$

III. Exponential Distribution of Asperity Heights

GW model first assume that the asperity height probability density follows a simple exponential distribution given by

$$_{\mathrm{simple}}\phi^*(z^*) = \begin{cases} 0 & \text{if } z^* < 0 \\ \mathrm{e}^{-z^*} & \text{if } z^* \geq 0 \end{cases} \tag{12}$$

Similarly, assume that the asperity height probability density follows another exponential distribution given by

$$_{\mathrm{generalized}}\phi^*(z^*) = \begin{cases} 0 & \text{if } z^* < 0 \\ a\,\mathrm{e}^{-bz^*} & \text{if } z^* \geq 0 \end{cases} \tag{13}$$

where a and b are positive constants. Note that the integral of $\int_{-\infty}^{+\infty} {}_{\mathrm{generalized}}\phi^*(z^*)\,\mathrm{d}z^* = a/b$ is equal to 1, only when $a = b$. Therefore strictly speaking, Eq. (13) is not an exponential probability density but only termed as the generalized one.

A. Elastic Contact of two Rough Surfaces

Substituting equation (13) into equations (1)-(3) yields

$$n = \frac{a}{b}\eta A_{\mathrm{n}}\,\mathrm{e}^{-bd^*} \tag{14}$$

$$A_{\mathrm{e}} = \frac{a}{b^2}\pi\beta A_{\mathrm{n}}\,\mathrm{e}^{-bd^*} \tag{15}$$

$$P_{\mathrm{e}} = \frac{a}{b^{2.5}}\sqrt{\pi}\beta E'\sqrt{\sigma/R}A_{\mathrm{n}}\,\mathrm{e}^{-bd^*} \tag{16}$$

Utilizing equation (16) yields

$$p_{\mathrm{n}}^* = \frac{p_{\mathrm{n}}}{\beta E'\sqrt{\sigma/R}} = \frac{a}{b^{2.5}}\sqrt{\pi}\,\mathrm{e}^{-bd^*} \tag{17}$$

Equation (16) dividing equation (15) gives

$$p_{\mathrm{e}} = \frac{P_{\mathrm{e}}}{A_{\mathrm{e}}} = \frac{1}{\sqrt{\pi b}}E'\sqrt{\frac{\sigma}{R}} \tag{18}$$

$$\frac{A_{\mathrm{e}}}{A_{\mathrm{n}}\beta} = \sqrt{\pi b}\frac{p_{\mathrm{n}}}{\beta E'\sqrt{\sigma/R}} \tag{19}$$

Equation (15) dividing equation (14) provides

$$\frac{A_{\mathrm{e}}}{nR\sigma} = \frac{\pi}{b} \tag{20}$$

B. Elastic-Plastic Contact of two Rough Surfaces

Inserting equation (13) into equations (5)-(9) gives

$$n^* = \frac{a}{b}\mathrm{e}^{-bd^*} \tag{21}$$

$$A_{\mathrm{e}}^* = \frac{a}{b^2}\pi\beta[1-(1+bw_{\mathrm{c}}^*)\,\mathrm{e}^{-bw_{\mathrm{c}}^*}]\mathrm{e}^{-bd^*} \tag{22}$$

$$A_{\mathrm{p}}^* = \frac{a}{b^2}\pi\beta(2+bw_{\mathrm{c}}^*)\,\mathrm{e}^{-bw_{\mathrm{c}}^*}\,\mathrm{e}^{-bd^*} \tag{23}$$

$$A^* = A_{\mathrm{e}}^* + A_{\mathrm{p}}^* = \frac{a}{b^2}\pi\beta(1+\mathrm{e}^{-bw_{\mathrm{c}}^*})\,\mathrm{e}^{-bd^*} \tag{24}$$

$$P_{\mathrm{e}}^* = \frac{2a}{b^{2.5}}\beta\sqrt{\pi\sigma/R}\times \left[\Phi\left(\sqrt{2bw_{\mathrm{c}}^*}\right) - 0.5 - \frac{3\sqrt{bw_{\mathrm{c}}^*}+2(bw_{\mathrm{c}}^*)^{1.5}}{3\sqrt{\pi}\,\mathrm{e}^{bw_{\mathrm{c}}^*}}\right]\mathrm{e}^{-bd^*} \tag{25}$$

$$P_{\mathrm{p}}^* = \frac{2a}{b^{2.5}}\beta\sqrt{\pi\sigma/R}\frac{6\sqrt{bw_{\mathrm{c}}^*}+3(bw_{\mathrm{c}}^*)^{1.5}}{3\sqrt{\pi}\,\mathrm{e}^{bw_{\mathrm{c}}^*}}\mathrm{e}^{-bd^*} \tag{26}$$

$$P^* = P_{\mathrm{e}}^* + P_{\mathrm{p}}^* = \frac{2a}{b^{2.5}}\beta\sqrt{\pi\sigma/R}\times \left[\Phi\left(\sqrt{2bw_{\mathrm{c}}^*}\right) - 0.5 + \frac{3\sqrt{bw_{\mathrm{c}}^*}+(bw_{\mathrm{c}}^*)^{1.5}}{3\sqrt{\pi}\,\mathrm{e}^{bw_{\mathrm{c}}^*}}\right]\mathrm{e}^{-bd^*} \tag{27}$$

where

$$\Phi(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{v^2}{2}}\,\mathrm{d}v \qquad (28)$$

is the normal probability integral and widely tabulated.

IV. Gaussian Distribution of Asperity Heights

Assume that the asperity height probability density follows the Gaussian distribution given by

$$_{\text{Gauss}}\phi^*(z^*)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{z^{*2}}{2}} \quad \text{if} -\infty<z^*<+\infty \qquad (29)$$

A. Elastic Contact of two Rough Surfaces

Introducing equation (29) into equations (1)-(3) gives

$$n^*=F_0(d^*)=1-\Phi(d^*) \qquad (30)$$

$$A_{\mathrm{e}}=\pi\beta A_{\mathrm{n}}F_1(d^*) \qquad (31)$$

$$P_{\mathrm{e}}=\frac{4}{3}\beta E'\sqrt{\sigma/R}A_{\mathrm{n}}F_{1.5}(d^*) \qquad (32)$$

$$p_{\mathrm{n}}^*=\frac{p_{\mathrm{n}}}{\beta E'\sqrt{\sigma/R}}=\frac{4}{3}F_{1.5}(d^*) \qquad (33)$$

where

$$F_i(d^*)=\int_{d^*}^{+\infty}(z^*-d^*)^i\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{z^{*2}}{2}}\,\mathrm{d}z^* \qquad (34)$$

Is a parabolic cylinder function whose values are given in Table I.

TABLE I: Some Discrete Values For Parabolic Cylinder Function

d^*	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$F_0(d^*)$	0.5	0.4602	0.4207	0.3821	0.3446	0.3085	0.2743	0.242	0.2119
$F_1(d^*)$	0.3989	0.3509	0.3069	0.2668	0.2304	0.1978	0.1687	0.1429	0.1202
$F_{1.5}(d^*)$	0.4299	0.3715	0.3191	0.2725	0.2313	0.1951	0.1636	0.1363	0.1127
d^*	0.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7
$F_0(d^*)$	0.1841	0.1587	0.1357	0.1151	0.0968	0.08076	0.06681	0.0548	0.04457
$F_1(d^*)$	0.1004	0.08332	0.06862	0.0561	0.04553	0.03667	0.0293	0.02324	0.01829
$F_{1.5}(d^*)$	0.09267	0.07567	0.06132	0.04935	0.03944	0.03129	0.02463	0.01925	0.01493
d^*	1.8	1.9	2	2.1	2.2	2.3	2.4	2.5	2.6
$F_0(d^*)$	0.03593	0.02872	0.02275	0.01786	0.0139	0.01072	0.008198	0.00621	0.004661
$F_1(d^*)$	0.01428	0.01105	0.00849	0.006468	0.004887	0.003662	0.00272	0.002004	0.001464
$F_{1.5}(d^*)$	0.01149	0.008773	0.006646	0.004995	0.003724	0.002754	0.00202	0.001469	0.00106
d^*	2.7	2.8	2.9	3	3.2	3.4	3.6	3.8	4
$F_0(d^*)$	0.003467	0.002555	0.001866	0.00135	0.0006871	0.0003369	0.0001591	0.00007235	0.00003167
$F_1(d^*)$	0.00106	0.0007611	0.0005417	0.0003822	0.0001852	0.00008666	0.00003911	0.00001702	0.000007145
$F_{1.5}(d^*)$	0.0007587	0.000538	0.0003784	0.0002639	0.0001251	0.00005724	0.00002529	0.00001079	0.000004438

Equation (32) dividing equation (31) gives

$$p_e = \frac{P_e}{A_e} = \frac{4F_{1.5}(d^*)}{3\pi F_1(d^*)} E' \sqrt{\frac{\sigma}{R}} \tag{35}$$

$$\frac{A_e}{A_n \beta} = \frac{3\pi F_1(d^*)}{4F_{1.5}(d^*)} \frac{p_n}{\beta E' \sqrt{\sigma / R}} \tag{36}$$

Equation (31) dividing equation (30) yields

$$\frac{A_e}{nR\sigma} = \pi \frac{F_1(d^*)}{F_0(d^*)} \tag{37}$$

B. Elastoplastic Contact of two Rough Surfaces

Putting equation (29) into equations (5)-(9) gives

$$n^* = 1 - \Phi(d^*) \tag{38}$$

$$A_e^* = \pi\beta[F_1(d^*) - F_1(d^* + w_c^*) - w_c^* F_0(d^* + w_c^*)] \tag{39}$$

$$A_p^* = \pi\beta[2F_1(d^* + w_c^*) + w_c^* F_0(d^* + w_c^*)] \tag{40}$$

$$A^* = A_e^* + A_p^* = \pi\beta[F_1(d^*) + F_1(d^* + w_c^*)] \tag{41}$$

$$P_e^* \approx \frac{4}{3}\beta\sqrt{\sigma / R}[F_{1.5}(d^*) - w_c^{*1.5} F_0(d^* + w_c^*)] \tag{42}$$

$$P_p^* = \frac{4}{3}\beta\sqrt{\sigma / R}[3\sqrt{w_c^*} F_1(d^* + w_c^*) + 1.5 w_c^{*1.5} F_0(d^* + w_c^*)] \tag{43}$$

$$P^* \approx P_e^* + P_p^* = \frac{4}{3}\beta\sqrt{\sigma / R}[F_{1.5}(d^*) + 3\sqrt{w_c^*} F_1(d^* + w_c^*) + 0.5 w_c^{*1.5} F_0(d^* + w_c^*)] \tag{44}$$

V. Some Fitting Examples

A. Elastic Contact of two Rough Surface

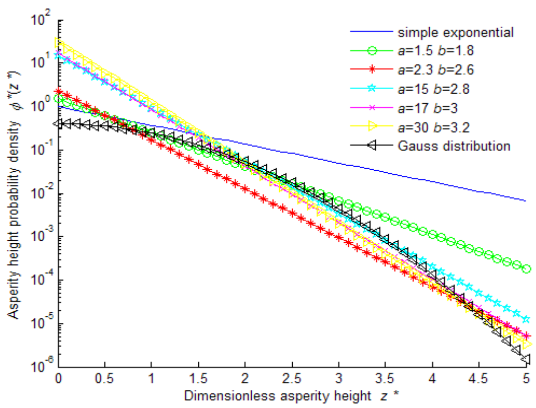

Fig. 1. Asperity height probability density versus dimensionless asperity height

Fig. 1 stands for the asperity height probability density. Contrary to the description made by GW model that "the exponential distribution is nevertheless a fair approximation to the uppermost 25% of the asperities of most surface", the simple exponential probability density is the worst fit to the Gaussian one especially at large z^*. The fourth generalized exponential probability density is a better fit to the Gaussian one in the large range of $1 \le z^* \le 5$. However, the first generalized exponential probability density is a better fit to that in the low range of $0 \le z^* \le 2$.

In the following calculations, $\eta = 300$ mm^{-2}, $R\sigma = 0.0001$ mm^2, $E'\sqrt{\sigma / R} = 25$ MPa, and $A_n = 1$ cm^2. Fig.2 denotes the elastic contact load. The fourth generalized exponential probability density is a better fit to the Gaussian one. However, the first generalized exponential probability density gives a better fit to that in the load range of $P_e \ge 10$ N as is already shown in the low range of $0 \le z^* \le 2$ in Fig.1. Contrary to the conclusion made by GW model that "the results approximate closely to those for the exponential distribution", the simple exponential probability density overestimates the surface separation at a certain contact load. At low $d^* = 0.4$ and applying the simple exponential probability density, first to fifth generalized one, Gaussian one, $P_e = 89.11$ N, 22.33 N, 9.914 N, 49.59 N, 43.66 N, 60.53 N, 23.13 N, respectively. Similarly, at large $d^* = 3.6$ and employing the simple exponential probability density, first to fifth generalized one, Gaussian one, $P_e = 3.632$ N, 0.07036 N, 0.002415 N, 0.00637 N, 0.002957 N, 0.002162 N, 0.002529 N, respectively.

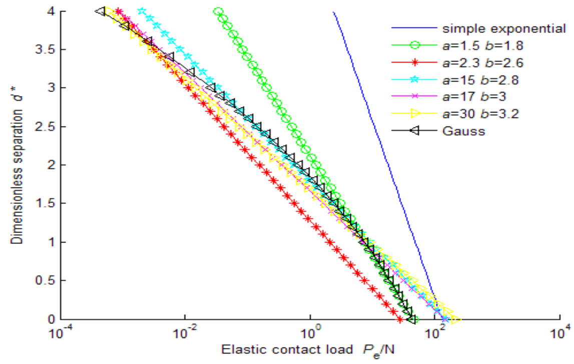

Fig. 2. Dimensionless separation versus elastic contact load

Fig. 3 gives the elastic contact area versus elastic contact load. The best agreement with the Gaussian result is the first generalized exponential probability density. The simple exponential probability density underestimates the elastic contact area at any given load.

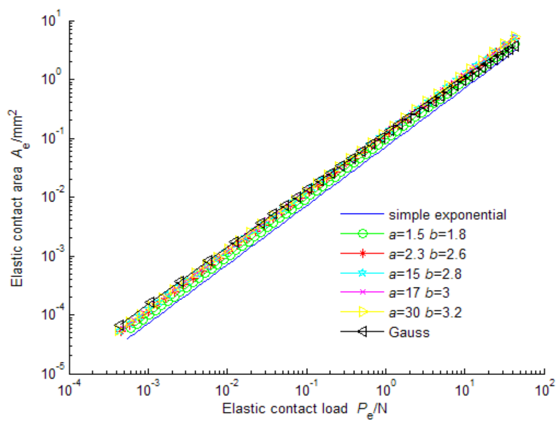

Fig. 3. Elastic contact area versus elastic contact load

Fig. 4 gives the elastic contact pressure versus elastic contact load. The simple exponential probability density overestimates the elastic contact pressure.

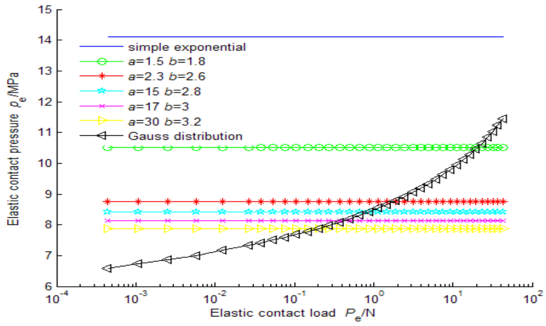

Fig. 4. Elastic contact pressure versus elastic contact load

Fig..5 provides the nominal pressure. The simple exponential probability density overestimates the surface separation at a given nominal pressure.

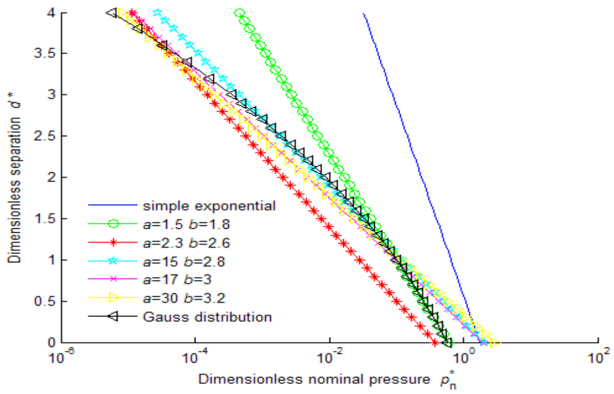

Fig. 5. Dimensionless separation versus dimensionless nominal pressure

B. *Elastoplastic Contact of two Rough Surfaces*

In the following calculations, $\beta = 0.078$, $w_c^* = 1$, $\sigma / R = 0.0039$, and $K = 0.577$. Fig.6 stands for the contact load. The simple exponential probability density overestimates the contact load at a given surface separation. The third generalized exponential probability density is the better fit to the Gaussian one.

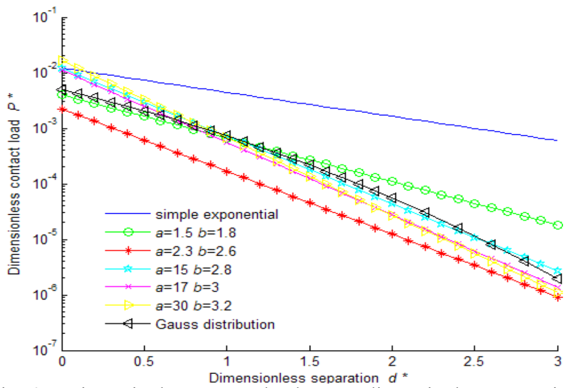

Fig. 6. Dimensionless contact load versus dimensionless separation

Fig.7 denotes the contact area. The simple exponential probability density also overestimates the contact area at a given surface separation. The third generalized exponential probability density is a better fit to the Gaussian one in a large range of surface separation while the first generalized exponential probability density is a better fit to that in a low range of surface separation.

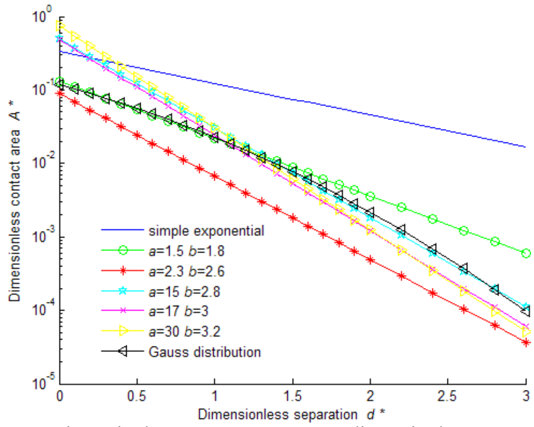

Fig. 7. Dimensionless contact area versus dimensionless separation

Fig. 8 denotes the contact spot number. The simple exponential probability density also overestimates the contact spot number at a given surface separation. However, the third to fifth generalized ones bring about unacceptable errors in the low range of surface separation.

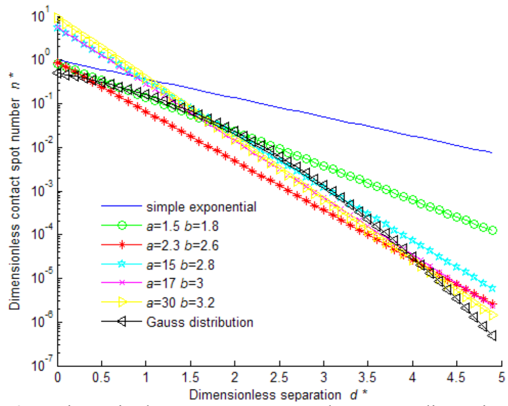

Fig. 8. Dimensionless contact spot number versus dimensionless separation

Fig. 9 denotes the plastic contact area. The simple exponential probability density also overestimates the plastic contact area at extremely large surface separation.

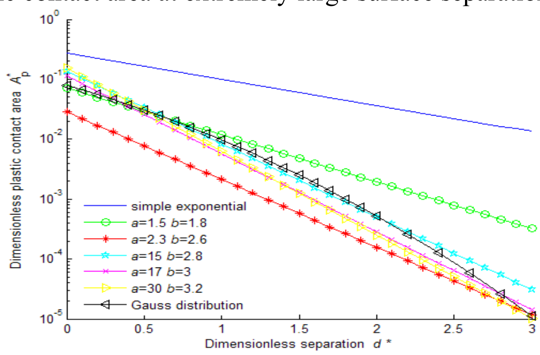

Fig. 9. Dimensionless plastic contact area versus dimensionless separation

VI. Comparison of Results for Various Contact Models

According to Eq. (13) and Figs. 1-9, a better generalized exponential probability density is chosen to fit the Gaussian one as follows

$$ {}_{\text{special}}\phi^*(z^*) = \begin{cases} 0 & \text{if } z^* < 0 \\ 17\,\mathrm{e}^{-3z^*} & \text{if } z^* \ge 0 \end{cases} \quad (45) $$

Bhushan solution for mean elastic real pressure was given below

$$\frac{{}_{\text{Bhushan}}p_{\text{e}}}{E'\sqrt{\sigma/R}} \approx 0.32 \tag{46}$$

Adopting equation (18) gives

$$\frac{{}_{\text{simple}}p_{\text{e}}}{E'\sqrt{\sigma/R}} = \frac{1}{\sqrt{\pi\times1}} = 0.5642 \tag{47}$$

$$\frac{{}_{\text{special}}p_{\text{e}}}{E'\sqrt{\sigma/R}} = \frac{1}{\sqrt{\pi\times3}} = 0.3257 \tag{48}$$

Applying equation (35) results in

$$\left.\frac{{}_{\text{Gauss}}p_{\text{e}}}{E'\sqrt{\sigma/R}}\right|_{d^*=3} = \frac{4F_{1.5}(d^*)}{3\pi F_1(d^*)} = \frac{4\times0.0002639}{3\pi\times0.0003822} = 0.293 \tag{49}$$

Bhushan solution for elastic real contact area was given below

$$\frac{{}_{\text{Bhushan}}A_{\text{e}}}{A_{\text{n}}\beta} = 3.2\frac{p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} \tag{50}$$

Utilizing equation (19) yields

$$\frac{{}_{\text{simple}}A_{\text{e}}}{A_{\text{n}}\beta} = \sqrt{\pi\times1}\frac{p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} = 1.7725\frac{p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} \tag{51}$$

$$\frac{{}_{\text{special}}A_{\text{e}}}{A_{\text{n}}\beta} = \sqrt{\pi\times3}\frac{p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} = 3.07\frac{p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} \tag{52}$$

It follows from equation (36) that

$$\begin{aligned}\left.\frac{{}_{\text{Gauss}}A_{\text{e}}}{A_{\text{n}}\beta}\right|_{d^*=3} &= \frac{3\pi F_1(d^*)}{4F_{1.5}(d^*)}\frac{p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} = \\ \frac{3\pi\times0.0003822}{4\times0.0002639}\frac{p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} &= 3.4124\frac{p_{\text{n}}}{\beta E'\sqrt{\sigma/R}}\end{aligned} \tag{53}$$

Bhushan solution for maximum elastic contact spot number was given below

$${}_{\text{Bhushan}}n^*\big|_{d^*=0} = 0.5 \tag{54}$$

Using equation (14) gives

$${}_{\text{simple}}n^*\big|_{d^*=0} = 1 \tag{55}$$

$${}_{\text{special}}n^*\big|_{d^*=0} = a/b = 1 \quad \text{when} \quad a=b \tag{56}$$

Adopting equation (30) gives

$${}_{\text{Gauss}}n^*\big|_{d^*=0} = 1-\Phi(d^*) = 1-0.5 = 0.5 \tag{57}$$

From Eqs. (54) and (57), the dimensionless maximum elastic contact spot number is 0.5 which shows the limitation of a surface with separation $-\infty < d^* < +\infty$ can be envisaged to be covered with a large number of protrusive asperities as well as excellent adjacent hollow valleys! However, from Eq. (56) the dimensionless maximum elastic contact spot number is 1 which shows the limitation of a surface with separation $d^* \geq 0$ can be envisaged to be absolute flat smooth.

Bhushan solution for elastic individual asperity real contact area was given below

$$\frac{{}_{\text{Bhushan}}A_{\text{e}}}{nR\sigma} \approx 1.21 \tag{58}$$

Applying equation (20) gives

$$\frac{{}_{\text{simple}}A_{\text{e}}}{nR\sigma} = \frac{\pi}{1} = 3.1416 \tag{59}$$

$$\frac{{}_{\text{special}}A_{\text{e}}}{nR\sigma} = \frac{\pi}{3} = 1.0472 \tag{60}$$

Utilizing equation (37) results in

$$\left.\frac{{}_{\text{Gauss}}A_{\text{e}}}{nR\sigma}\right|_{d^*=3} = \pi\frac{F_1(d^*)}{F_0(d^*)} = \pi\frac{0.0003822}{0.00135} = 0.8894 \tag{61}$$

CEB model solution for mean plastic real pressure was given below

$${}_{\text{CEB}}p_{\text{p}} = KH \tag{62}$$

Equation (26) dividing equation (23) gives

$${}_{\text{simple}}p_{\text{p}} = {}_{\text{special}}\,p_{\text{p}} = P_{\text{p}}/A_{\text{p}} = KH \tag{63}$$

Equation (43) dividing equation (40) gives

$${}_{\text{Gauss}}p_{\text{p}} = P_{\text{p}}/A_{\text{p}} = KH \tag{64}$$

Bhushan solution for elastic nominal pressure was given below

$$\frac{{}_{\text{Bhushan}}p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} \leq 0.57 \tag{65}$$

Applying equation (17) provides

$$\frac{{}_{\text{simple}}p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} = \sqrt{\pi}\,\mathrm{e}^{-d^*} \leq 1.7725 \tag{66}$$

$$\frac{{}_{\text{special}}p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} = \frac{17}{3^{2.5}}\sqrt{\pi}\,\mathrm{e}^{-3d^*} \leq 1.933 \tag{67}$$

Using equation (33) obtains

$$\frac{{}_{\text{Gauss}}p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} = \frac{4}{3}F_{1.5}(d^*) \leq \frac{4}{3}\times0.4299 = 0.5732 \tag{68}$$

Employing equation (50) gives

$${}_{\text{Bhushan}}A_{\text{e}}^* = 3.2\beta\frac{{}_{\text{Bhushan}}p_{\text{n}}}{\beta E'\sqrt{\sigma/R}} \leq 3.2\times0.078\times0.57 = 0.1423 \tag{69}$$

Using equation (15) gives

$${}_{\text{simple}}A_{\text{e}}^* = \pi\beta\,\mathrm{e}^{-d^*} \leq \pi\times0.078 = 0.245 \tag{70}$$

$${}_{\text{special}}A_{\text{e}}^* = \frac{17}{3^2}\pi\beta\,\mathrm{e}^{-3d^*} \leq \frac{17}{9}\pi\times0.078 = 0.4629 \tag{71}$$

Using equation (31) results in

$${}_{\text{Gauss}}A_{\text{e}}^* = \pi\beta F_1(d^*) \leq \pi\times0.078\times0.3989 = 0.09775 \tag{72}$$

GW model criterion for the onset of a significant degree of plasticity was $\psi = 1$ or $w_{\text{e}}^* = 1$ or another form given below

$$\frac{{}_{\mathrm{GW}}A_{\mathrm{p}}^{*}}{A^{*}} = 0.02 \quad (73)$$

Equation (23) dividing equation (24) gives

$$\left.\frac{{}_{\mathrm{simple}}A_{\mathrm{p}}^{*}}{A^{*}}\right|_{w_{\mathrm{c}}^{*}=1} = \frac{2+w_{\mathrm{c}}^{*}}{1+\mathrm{e}^{w_{\mathrm{c}}^{*}}} = \frac{3}{1+\mathrm{e}} = 0.8068 \quad (74)$$

$$\left.\frac{{}_{\mathrm{special}}A_{\mathrm{p}}^{*}}{A^{*}}\right|_{w_{\mathrm{c}}^{*}=1} = \frac{2+3w_{\mathrm{c}}^{*}}{1+\mathrm{e}^{3w_{\mathrm{c}}^{*}}} = \frac{5}{1+\mathrm{e}^{3}} = 0.2371 \quad (75)$$

Equation (40) dividing equation (41) provides

$$\left.\frac{{}_{\mathrm{Gauss}}A_{\mathrm{p}}^{*}}{A^{*}}\right|_{w_{\mathrm{c}}^{*}=1,d^{*}=3} = \frac{2F_{1}(d^{*}+w_{\mathrm{c}}^{*})+w_{\mathrm{c}}^{*}F_{0}(d^{*}+w_{\mathrm{c}}^{*})}{F_{1}(d^{*})+F_{1}(d^{*}+w_{\mathrm{c}}^{*})} = \frac{2\times 0.000007145+0.00003167}{0.0003822+0.000007145} = 0.118 \quad (76)$$

Pullen-Williamson [20] solution for plastic real contact area for the onset of plastic deformation was given below

$$\left.{}_{\mathrm{PW}}A_{\mathrm{p}}^{*}\right|_{w_{\mathrm{c}}^{*}=1} \le 0.1425\beta\pi K/\sqrt{w_{\mathrm{c}}^{*}} = 0.1425\times 0.078\pi\times 0.577 = 0.02015 \quad (77)$$

Applying equation (23) gives

$$\left.{}_{\mathrm{simple}}A_{\mathrm{p}}^{*}\right|_{w_{\mathrm{c}}^{*}=1} \le \pi\beta(2+w_{\mathrm{c}}^{*})/\mathrm{e}^{w_{\mathrm{c}}^{*}} = \pi\times 0.078\times 3/\mathrm{e} = 0.2704 \quad (78)$$

$$\left.{}_{\mathrm{special}}A_{\mathrm{p}}^{*}\right|_{w_{\mathrm{c}}^{*}=1} \le \frac{17}{3^{2}}\pi\beta(2+3w_{\mathrm{c}}^{*})/\mathrm{e}^{3w_{\mathrm{c}}^{*}} = \frac{17}{9}\pi\times 0.078\times 5/\mathrm{e}^{3} = 0.1152 \quad (79)$$

Using equation (40) yields

$$\left.{}_{\mathrm{Gauss}}A_{\mathrm{p}}^{*}\right|_{w_{\mathrm{c}}^{*}=1} = \pi\beta[2F_{1}(d^{*}+1)+F_{0}(d^{*}+1)] \le \pi\times 0.078\times(2\times 0.08332+0.1587) = 0.07972 \quad (80)$$

Finally, the practical solutions of some parameters for the elastic contact and elastoplastic contact and main digital features of the two contacting rough surfaces are given in Table II.

TABLE II: PRACTICAL SOLUTIONS FOR ELASTIC AND ELASTOPLASTIC CONTACT [21,22]

parameter	elastic contact	elastoplastic contact
surface separation	$d^{*} = 1.4\lg^{0.65}\dfrac{0.5732\beta E'\sqrt{\sigma/R}}{p_{\mathrm{n}}}$	——
maximum nominal pressure	$p_{\mathrm{n\,max}} = 0.5732\beta E'\sqrt{\sigma/R}$	——
contact pressure	$\dfrac{p_{\mathrm{e}}}{E'\sqrt{\sigma/R}} = 0.42\left(\dfrac{p_{\mathrm{n}}}{\beta E'\sqrt{\sigma/R}}\right)^{0.04} \approx 0.3229$	$p_{\mathrm{p}} = KH$
maximum contact pressure	$p_{\mathrm{e\,max}} = 0.4574E'\sqrt{\sigma/R}$	$p_{\mathrm{p\,max}} = 0.659H$
contact area	$\dfrac{A_{\mathrm{e}}^{*}}{\beta} = 2.4\left(\dfrac{p_{\mathrm{n}}}{\beta E'\sqrt{\sigma/R}}\right)^{0.96} \approx \dfrac{3.2p_{\mathrm{n}}}{\beta E'\sqrt{\sigma/R}}$	$A_{\mathrm{p}}^{*} = \dfrac{p_{\mathrm{n}}}{2H}$
maximum contact area	$A_{\mathrm{e\,max}}^{*} = 0.09775$	$A_{\mathrm{p\,max}}^{*} = 0.07972$
contact spot number	$n^{*} = 1.2\left(\dfrac{p_{\mathrm{n}}}{\beta E'\sqrt{\sigma/R}}\right)^{0.88} \approx \dfrac{2.6446p_{\mathrm{n}}}{\beta E'\sqrt{\sigma/R}}$	$n \approx \dfrac{57500\sqrt{2}}{\sqrt{\eta}R}\dfrac{E'}{H}\left(\dfrac{p_{\mathrm{n}}}{H}\right)^{0.91}$
maximum spot number	$n_{\max}^{*} = 0.5$	$n_{\max}^{*} = 0.5$
an asperity contact area	$\dfrac{A_{\mathrm{e}}}{nR\sigma} = 2\left(\dfrac{p_{\mathrm{n}}}{\beta E'\sqrt{\sigma/R}}\right)^{0.08} \approx 1.21$	$\dfrac{A_{\mathrm{p}}^{*}}{n} \approx 0.000004348\sqrt{2\eta}R\dfrac{H}{E'}\left(\dfrac{p_{\mathrm{n}}}{H}\right)^{0.09}$

VII. CONCLUSIONS

Some practical analytic solutions were deduced for the contact load, contact area and contact spot number for both GW elastic contact model and CEB elastoplastic contact model by the generalized exponential probability density fitting the Gaussian one. The simple exponential probability density overestimates the contact load, contact area and contact spot number at a certain surface separation. In general, when the generalized exponential probability density was used to fit the Gaussian one, some solutions are in good agreement with the Gaussian numerical solutions. Finally, some digital features of two contacting rough surfaces were given by comparing different contact models.

The natural logarithmic expressions of Eqs. (13) and (29) are $\ln a - bz^{*}$ and $-0.5z^{*2} - 0.5\ln(2\pi)$, respectively. Therefore, approximate fitting essentially refers to choose a certain line replacing the explicit parabolic line. a denotes the amplitude of probability density and b the magnitude of surface separation. For $d^{*} \gg 3$, the roughness effects are not important and smooth film theory is sufficiently accurate. When d^{*} approaches 3, the roughness effects become important. As d^{*} decreases further, asperities start interacting with each other and contacts form. Consequently, the range of $d^{*} < 3$ is called as the partial lubrication regime where the effect of roughness is most important. The surface separation is not very sensitive to the nominal pressure: in fact the mean planes of two similar surfaces in contact are usually separated by σ to 2σ. In words, Eq.

(45) replacing Eq. (29) can reflect the main features of the two contacting rough surfaces in the partial lubrication regime especially for $1 < d^* < 3$.

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