Solving Multi-Object Programming Problems by Homotopy Inner Point Method under Quasi-Normal Cone Condition

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Abstract—**This paper, we discussed a general multi-objective optimization problem under quasi-normal cone condition.we prove that for almost every point in the feasible region, a smooth and bounded homotopy path can be derived and that the algorithm converges to the K-K-T point of multi-objective programming problem. Numerical simulation confirmed the viability of this method.**

***Index Terms*—multi-objective programming; homotopy method; global convergence.**

I. INTRODUCTION

We consider the following multi-objective programming problem

$$(MOP1)\quad \begin{cases} \min f(x) \\ \text{s.t } h(x) \le 0 \\ g(x) = 0 \end{cases}$$

Where $x \in R^n$,

$$f = (f_1, f_2, \cdots, f_p)^T,$$
$$h = (h_1, h_2, \cdots, h_l)^T,$$
$$g = (g_1, g_2, \cdots, g_m)^T$$

are three continuously differentiable functions.

Let $\overline{\Omega} = \{x \in R^n \mid g(x) = 0, h(x) \le 0\}$ be the feasible set, and $\Omega = \{x \in R^n \mid g(x) = 0, h(x) < 0\}$ be the strictly feasible set.

$I(x) = \{\gamma \in \{1,\ 2,\ \cdots, l\} \big| h_\gamma(x) = 0\}$ is the active index set,

$$x \circ y = (x_1 y_1,\ x_2 y_2,\ \cdots,\ x_n y_n)^T,$$

$$\Lambda^+ = \{\lambda \in R^P \mid \lambda_i \ge 0,\ i = 1, 2, \cdots, p,\ \sum_{i=1}^{p} \lambda_i = 1\},$$

$$\Lambda^{++} = \{\lambda \in R^P \mid \lambda_i > 0,\ i = 1, 2,\ \cdots,\ p,\ \sum_{i=1}^{p} \lambda_i = 1\},$$

$$\nabla f(x) \in R^{n \times p},\ \nabla h(x) \in R^{n \times l},\ \nabla g(x) \in R^{n \times m}.$$

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Denote the Jacobi matrix of $f(x), h(x), g(x)$, respectively.

$$P = \{1, 2, \cdots, p\},\ L = \{1,\ 2, \cdots,\ l\},\ M = \{1,\ 2,\ \cdots,\ m\},$$
$$e = (1,\ 1,\ \cdots,\ 1)^T \in R^p.$$

Instead of (MOP1) problem, we consider an associated non-convex nonlinear scalar optimization problem as follows:

$$(P_2)\quad \begin{cases} \min \lambda^T f(x) \\ \text{s.t } h(x) \le 0 \\ g(x) = 0 \\ \lambda \in \Lambda^+ \end{cases}$$

Definition 1.1 If (x^*, λ^*) is a optimistic solutions of P_2, then x^* is minimal weak efficient solution of (MOP1)[1].

Let x, λ be a KKT point of P_2, Then there exist $y \in R^m$, $z \in R^l_+$, $\xi \in R^p_+$, and $h \in R$, such that

$$\begin{cases} \nabla f(x)\lambda + \nabla g(x) y + \nabla h(x) z = 0 \\ f(x) - \xi - h \cdot e = 0 \\ z \circ h(x) = 0,\ \ h(x) \le 0,\ \ z \ge 0 \\ g(x) = 0 \\ \xi \circ \lambda = 0\ \ \lambda \ge 0,\ \ \xi \ge 0 \\ 1 - \sum_{i=1}^{p} \lambda_i = 0 \end{cases} \tag{1.1}$$

Therefore, to find a minimal weak efficient solution, we have to solve problem (1.1).

The homotopy method is an important globally convergence method. In 1988, Megiddo [2] and Kojima et al. [3] discovered that the attractive Karmarkar interior point method for linear programming was a kind of path-following method. The homotopy method for mathematical programming has become an active research field. Reference [4-6] presented a new interior point method—combined homotopy interior point method (CHIP method) for nonlinear programming with inequality constrain under a certain contions. Reference [7-8] enlarged to equality constrain under cone condition and quasi-normal cone condition.

Recently Z.H.Lin and Z.P.Sheng [9]generalized the CHIP method to convex multi-objective programming (MOP) with only inequality constraints. In 2008, W.Song and G.M.Yao [10] generalized the CHIP method to the general (MOP) under so-called normal cone condition. In this paper, we extended the method in ref. [8] to (MOP) By adopting it's concept of independent positive irrelative, we provided an quasi-normal cone condition that was weaker than that in ref. [10] and construct a new combined homotopy mapping which is much different from that in [10].This paper is organized as follows: Section 2 we recall some notations and preliminaries results. In Section 3, we establish the combined homotopy equation. The existence and convergence of a smooth homotopy path from almost any initial point $w^{(0)}$ to a solution of the KKT system of (MOP) are proved. Finally, a numerical algorithm is given, and a numerical example shows that this method is feasible and effective in Section 4.

II. PRELELIMINARIES

Definition2.1[8] Suppose that Ω is nonempty, there exist twice continuously differentiable mappig $\alpha_r : R^n \to R^n$, $(r = 1,2,\cdots,l)$ and $\beta : R^n \to R^{n\times m}$. For any $x \in \overline{\Omega}$, $\{\beta(x), \alpha_r(x) : r \in I(x)\}$ is said to be positive independently irrelative with respect to $\nabla h(x)$, if

$$\beta(x)y + \sum_{r\in I(x)} (z_r \nabla h_r(x) + u_r \alpha_r(x)) = 0,\ y \in \mathrm{R}^m, z_r \geq 0, u_r \geq 0$$

implies that $y = 0, z_r = 0, u_r = 0\ \ (r \in I(x))$.

The following are four assumptions used in the literature:

(H1) Ω is nonempty and bounded;

(H2) $\forall x \in \overline{\Omega}$ and $t \in [0,1]$, $(\nabla g(x) + t(\beta(x) - \nabla g(x))$ is positive independently irrelative with respect to $\nabla h(x)$

(H3) $\forall x \in \overline{\Omega}$,we have

$$\left\{x + \sum_{r\in I(x)} z_r \alpha_r(x) + \beta(x)y :\ y \in \mathrm{R}^m, z_r \geq 0, for\ \ r \in I(x)\right\} \bigcap \overline{\Omega} = x$$

(H4) $\forall x \in \overline{\Omega}$, $\nabla g(x)$ is matrix of full column rank and $\nabla^T g(x)\beta(x)$ is nonsingular

If we choose $\alpha(x) = \nabla h(x)$, $\beta(x) = \nabla g(x)$, then it is easy to get Ω satisfies the normal cone condition of [10]; By the converse does not hold, this can be illustrated by the simple example at the end of this paper .So enlarge the use value .

Definition 2.2 Let M, N be differential manifolds with $\dim N = p$ and let $H : M \to N$ be a differentiable mapping. If $\mathrm{rank}(\partial H(x)/\partial x) = p,\ \forall x \in H^{-1}(y)$, we say that that $y \in N$ is a regular value of H and $x \in M$ is a regular point. Given a curve $\Gamma \subset H^{-1}(y)$, if every $x \in \Gamma$ is a regular point, then we say that Γ is a regular path.

Lemma 2.1 (Parametric Form of the Sard Theorem on a Manifold with Boundary, see [11]) Let Q and N be differential manifolds of dimension q and p, respectively. Let M be a m-dimensional differential manifold with boundary. Suppose that $\Phi : Q \times M \to N$ is a C^r mapping, where $r > \max\{0, m - p\}$. If $0 \in N$ is a regular value of Φ and $\partial\Phi$, then for almost all $a \in Q$, 0 is a regular value of $\Phi_a = \Phi(a,\cdot)$ and $\partial\Phi_a$, where $\partial\Phi$, $\partial\Phi_a$ denote the restriction of Φ and Φ_a to $Q \times \partial M$ and ∂M, respectively.

Lemma 2.2 (Inverse Image Theorem, See [11,12]) Suppose that M is a $m-$ dimensional C^r differential manifold with boundary, N is a $p-$ dimensional C^r differential manifold, $r \geq 1$, and $\Phi : M \to N$ is a C^r map. If $a \in N$ is a regular value of Φ and $\partial\Phi$ then either $S = \Phi^{-1}(a)$ is empty or a $(m-p)$ dimensional submanifold, and $\partial S = S \bigcap \partial M$.

Lemma 2.3 (classification theorem of one-dimensional manifold with boundary, see [12]) Each connected part of a one-dimensional manifold with boundary is homeomorphic either to a unit circle or to a unit interval.

III. MAIN RESULTS

Now we construct a homotopy mapping as follows:

$$H : \Omega \times \Lambda^{++} \times R^m \times R^{l+p}_{++} \times R \times (0,1] \to R^{n+m+l+2p+1}$$

$$H(w, w^{(0)}, t) = \begin{bmatrix} (1-t)\left[\nabla f(x)\lambda + \nabla h(x)z + t\alpha(x)z^2\right] + \\ \left[\nabla g(x) + t(\beta(x) - \nabla g(x))\right]y + t(x - x^{(0)}) \\ (1-t)\left[f(x) - \xi\right] - (h - h^{(0)}) \cdot e + t(\lambda - \lambda^{(0)}) \\ z \circ h(x) - tz^{(0)} \circ h(x^{(0)}) \\ g(x) \\ \xi \circ \lambda - t\xi^{(0)} \circ \lambda^{(0)} \\ 1 - \sum_{i=1}^{p} \lambda_i \end{bmatrix} = 0 \tag{3.1}$$

Let

$$w = (x, \lambda, y, z, \xi, h) \in R^{n+m+l+2p+1},$$

$$w^{(0)} = (x^{(0)}, \lambda^{(0)}, y^{(0)}, z^{(0)}, \xi^{(0)}, h^{(0)}) \in \Omega \times \Lambda^{++} \times R^m \times R^{l+p}_{++} \times \{0\}.$$

When $t = 1$, the homotopy equation (3.1) becomes

$$\begin{cases} \beta(x)y + x - x^{(0)} = 0 & (3.2a) \\ -(h - h^{(0)}) \cdot e + (\lambda - \lambda^{(0)}) = 0 & (3.2b) \\ z \circ h(x) - z^{(0)} \circ h(x^{(0)}) = 0 & (3.2c) \\ g(x) = 0 & (3.2d) \\ \xi \circ \lambda - \xi^{(0)} \circ \lambda^{(0)} = 0 & (3.2e) \\ 1 - \sum_{i=1}^{p} \lambda_i = 0 & (3.2f) \end{cases}$$

Using assumption (H3) , the equation of (3.2a) and (3.2d) , we have $x = x^{(0)} \in \Omega$, By the Assumption (H2) and (3.2c), we get $y = y^{(0)} = 0$ and $z = z^{(0)}$ .Since $\sum_{i=1}^{p} \lambda_i^{(0)} = 1$, $\lambda^{(0)} > 0$, $h^{(0)} = 0$, we have $\lambda = \lambda^{(0)}$, $\xi = \xi^{(0)}$, $h = h^{(0)} = 0$. Thus, the equation $H(w, w^{(0)}, 1) = 0$ has only one solution $w = w^{(0)}$.

When $t = 0$, $H(w, w^{(0)}, t) = 0$ turns out to be problem (1.1). For a given $w^{(0)}$, rewrite $H(w, w^{(0)}, t)$ as $H_{w^{(0)}}(w,t)$.

The zero-point set of $H_{w^{(0)}}$ is

$$H^{-1}_{w^{(0)}}(0) = \{(w,t) \mid H_{w^{(0)}}(w,t) = 0\}.$$

Since $H(w^{(0)}, w^{(0)}, t) = 0$, we have $H^{-1}_{w^{(0)}}(0) \neq \phi$.

Lemma 3.1 Let H be defined as (3.1), $f_i(x)(i\in P)$, $h_\gamma(x)(\gamma\in L)$, $g_j(x)(j\in M)$ are three times continuously differentiable functions, and let the conditions (H1)-(H4) hold. Then for almost all $w^{(0)}\in\Omega\times\Lambda^{++}\times R^m\times R^{l+p}_{++}\times\{0\}$, 0 is a regular value of $H(w,w^{(0)},t)$, and $H^{-1}_{w^{(0)}}(0)$ consists of some smooth curves. Among them, a smooth curve, say $\Gamma_{w^{(0)}}$, is starting from $(w^{(0)},1)$.

Proof: Denote the Jacobi matrix of $H(w,w^{(0)},t)$ by $H^{'}(w,w^{(0)},t)$.

For any $w^{(0)}\in\Omega\times\Lambda^{++}\times R^m\times R^{l+p}_{++}\times\{0\}$ and $t\in(0,1]$, we have

$$\frac{\partial H(w,w^{(0)},t)}{\partial(x,x^{(0)},\lambda_1,\lambda^{(0)},z^{(0)},\xi^{(0)})}=$$

$$\begin{pmatrix} Q(x) & -tE_n & (1-t)\nabla f_1(x) & 0 & 0 & 0\\ (1-t)\nabla^T f(x) & 0 & te_p & -tE_p & 0 & 0\\ z\nabla^T h(x) & -tz^{(0)}\nabla^T h(x^{(0)}) & 0 & 0 & -tZ & 0\\ \nabla^T g(x) & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \xi_1 e_p & -t\theta & 0 & -t\Lambda\\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{3.3}$$

Where

$$Q(x)=(1-t)\left[\sum_{i=1}^{p}\lambda_i\nabla^2 f_i(x)+\sum_{\gamma=1}^{l}z_\gamma\nabla^2 h_\gamma(x)+t\sum_{r=1}^{l}z_\gamma^2\nabla\alpha_r(x)\right]$$
$$+\sum_{j=1}^{m}(1-t)y_j\nabla^2 g_j(x)+\sum_{j=1}^{m}y_j\nabla\beta_j(x)+tE_n,$$
$$\theta=\mathrm{diag}(\xi^{(0)}),\ Z=\mathrm{diag}(h(x^{(0)})),$$
$$\Lambda=\mathrm{diag}(\lambda^{(0)}),\ e_p=(1,0,\cdots;0)^T\in R^p.$$

$\partial H(w,w^{(0)},t)/\partial(x,x^{(0)},\lambda_1,\lambda^{(0)},z^{(0)},\xi^{(0)})$ is a matrix of full row rank, because $\lambda_i^{(0)}>0,(i\in P)$, $h_\gamma(x^{(0)})<0,(\gamma\in L)$ and $\nabla g(x)$ are matrices of full column rank. That is, 0 is a regular value of $H(w,w^{(0)},t)$. By the Parameterized Sard Theorem on smooth manifold,lemma 2.1, for almost all $w^{(0)}\in\Omega\times\Lambda^{++}\times R^m\times R^{l+p}_{++}\times\{0\}$, 0 is a regular value of mapping $H_{w^{(0)}}$, By the inverse image theorem lemma 2.2, $H^{-1}_{w^{(0)}}(0)$ consists of some smooth curves. Because $H_{w^{(0)}}(w^{(0)},1)=0$, there must be a smooth curve $\Gamma_{w^{(0)}}$ starting from $(w^{(0)},1)$.

Lemma 3.2 Conditions of Lemma 3.1 are satisfied. For a given $w^{(0)}\in\Omega\times\Lambda^{++}\times R^m\times R^{l+p}_{++}\times\{0\}$, 0 is a regular value of $H_{w^{(0)}}$, then in $\Omega\times\Lambda^{++}\times R^m\times R^{l+p}_{++}\times R\times(0,1]$, $\Gamma_{w^{(0)}}$ is a bounded curve.

Proof: From (3.1), it is easy to see that $\Gamma_{w^{(0)}}\subset\Omega\times\Lambda^{++}\times R^m\times R^{l+p}_{++}\times R\times(0,1]$. If $\Gamma_{w^{(0)}}$ is an unbounded curve, then there exists a sequence of points $\{(w^{(k)},t_k)\}\subset\Gamma_{w^{(0)}}$ such that $\|(w^{(k)},t_k)\|\to\infty$. Because $\Omega\times\Lambda^{++}$ and $t\in(0,1]$ are bounded sets, therefore there exists a subsequence of point $\{(w^{(k_i)},t_{k_i})\}$ (for brevity, we will use k instead of k_i in the rest part of this paper) such that $x^{(k)}\to\bar{x}\in\bar{\Omega}$, $\lambda^{(k)}\to\bar{\lambda}\in\Lambda^+$, $t_k\to\bar{t}\in[0,1]$, and

$$\left\|\left(y^{(k)},z^{(k)},\xi^{(k)},h^{(k)}\right)\right\|\to\infty,\ \text{as } k\to\infty. \tag{3.4}$$

1) From the second.fifth and sixth equality of (3.1), we have

$$(1-t_k)\left[f(x^{(k)})-\xi^{(k)}\right]-h^{(k)}e+t_k(\lambda^{(k)}-\lambda^{(0)})=0 \tag{3.5}$$
$$\xi^{(k)}\circ\lambda^{(k)}=t_k\xi^{(0)}\circ\lambda^{(0)}, \tag{3.6}$$
$$\sum_{i=1}^{p}\lambda_i=1 \tag{3.7}$$

So
$$\begin{aligned}&(1-t_k)\left[(\lambda^{(k)})^T f(x^{(k)})-t_k(\xi^{(0)})^T\lambda^{(0)}\right]\\ &-h^{(k)}+t_k\left[(\lambda^{(k)})^T\lambda^{(k)}-(\lambda^{(k)})^T\lambda^{(0)}\right]=0\end{aligned} \tag{3.8}$$

as $k\to\infty$, the first and third parts at the left-hand side of (3.8) are bounded.thus $h^{(k)}$ is finity, $h^{(k)}\to\bar{h}$.

2) if $\|\xi^{(k)}\|\to\infty$, let $I_1=\{i\in P|\lim_{k\to\infty}\xi_i^{(k)}=+\infty\}$, from the fifth equation of (3.1), $I_1\subset I_1(\bar{\lambda})\triangleq\{i\in P/\bar{\lambda}_i=0\}$ and $I_1\neq\phi$.

For $\bar{t}<1$, from (3.5), we have

$$\begin{aligned}&\lim_{k\to\infty}\left\{(1-t_k)\left[f(x^{(k)})-\xi^{(k)}\right]-h^{(k)}e+t_k(\lambda^{(k)}-\lambda^{(0)})\right\}\\ &=(1-\bar{t})f(\bar{x})-(1-\bar{t})\lim_{k\to\infty}\xi^{(k)}-\bar{h}e+\bar{t}(\bar{\lambda}-\lambda^{(0)})=0\end{aligned}$$

So $\lim_{k\to\infty}\xi^{(k)}=\bar{\xi}\neq\infty$,

For $\bar{t}=1$, from (3.5) and (3.7) we have

$$-\lim_{k\to\infty}\left[(1-t_k)\xi_i^{(k)}\right]-\bar{h}-\lambda_i^{(0)}=0\quad i\in I_1(\bar{\lambda}) \tag{3.9}$$
$$-\bar{h}+(\bar{\lambda}_i-\lambda_i^{(0)})=0\quad i\notin I_1(\bar{\lambda}) \tag{3.10}$$
$$1-\sum_{i\notin I_1(\bar{\lambda})}\bar{\lambda}_i=0 \tag{3.11}$$

From (3.10) and (3.11) we have

$$l\bar{h}=1-\sum_{i\notin I_1(\bar{\lambda})}\lambda_i^{(0)}>0,\quad l=p-\#I_1(\bar{\lambda}) \tag{3.12}$$

From (3.9),we notice $\xi_i^{(k)}>0,(1-t_k)>0,\lambda_i^{(0)}>0$
Hence

$$0\geq-\lim_{k\to\infty}(1-t_k)\xi_i^{(k)}=\bar{h}+\lambda_i^{(0)}>0\quad i\in I_1(\bar{\lambda}) \tag{3.13}$$

It is contradiction with (3.13), thus $\|\xi^{(k)}\|\to\infty$ is impossible.

3) From the first equality of (3.1), we have

$$\begin{aligned}&(1-t_k)\left[\nabla f(x^{(k)})\lambda^{(k)}+\nabla h(x^{(k)})z^{(k)}+t_k\alpha(x^{(k)})(z^{(k)})^2\right]+\\ &\left[\nabla g(x^{(k)})+t_k(\beta(x^{(k)})-\nabla g(x^{(k)}))\right]y^{(k)}+t_k(x^{(k)}-x^{(0)})=0\end{aligned} \tag{3.14}$$

Rewrite (3.14) as

$$\begin{aligned}&\sum_{\gamma\in I(\bar{x})}(1-t_k)\left[z_\gamma^{(k)}\nabla h_\gamma(x^{(k)})+t_k(z_r^{(k)})^2\alpha_r(x^{(k)})\right]\\&+\left[\nabla g(x^{(k)})+t_k(\beta(x^{(k)})-\nabla g(x^{(k)}))\right]y^{(k)}\\&=(1-t_k)\big[-\sum_{\gamma\notin I(\bar{x})}(z_\gamma^{(k)}\nabla h_\gamma(x^{(k)})+t_k(z_r^{(k)})^2\alpha_r(x^{(k)}))\\&-\nabla f(x^{(k)})\lambda^{(k)}\big]-t_k(x^{(k)}-x^{(0)})\end{aligned}\tag{3.15}$$

As $k\to\infty$, the right side of (3.15) is bounded, By the Assumption (H2),we conclude $\|y^{(k)}\|\to\infty$ is impossible.

So that $y^{(k)}\to\bar{y}\in R^m$

4) if $\|z^{(k)}\|\to\infty$, let $I=\{\gamma\in L|\lim_{k\to\infty}z_\gamma^{(k)}=+\infty\}$, from the third equation of (3.1), that $I\subset I(\bar{x})$ and $I\neq\phi$.

For $\bar{t}<1$, rewrite (3.14) as

$$\begin{aligned}&\sum_{\gamma\in I(\bar{x})}(1-t_k)\left[z_\gamma^{(k)}\nabla h_\gamma(x^{(k)})+t_k(z_r^{(k)})^2\alpha_r(x^{(k)})\right]\\&+(1-t_k)\big[\nabla f(x^{(k)})\lambda^{(k)}+\sum_{\gamma\notin I(\bar{x})}(z_\gamma^{(k)}\nabla h_\gamma(x^{(k)})\\&+t_k(z_r^{(k)})^2\alpha_r(x^{(k)}))\big]+\big[\nabla g(x^{(k)})+t_k(\beta(x^{(k)})\\&-\nabla g(x^{(k)}))\big]y^{(k)}+t_k(x^{(k)}-x^{(0)})=0\end{aligned}\tag{3.16}$$

From $z_\gamma^{(k)}\to\infty, r\in I(\bar{x})$ and the conditions (H2), as $k\to\infty$, the first part at the left-hand side of (3.16) tends to infinity, but the second, third and fourth parts are bounded. This is impossible.

For $\bar{t}=1$, rewrite (3.14) as

$$\begin{aligned}&\sum_{\gamma\in I(\bar{x})}(1-t_k)\left[z_\gamma^{(k)}\nabla h_\gamma(x^{(k)})+t_k(z_r^{(k)})^2\alpha_r(x^{(k)})\right]\\&+\left[\nabla g(x^{(k)})+t_k(\beta(x^{(k)})-\nabla g(x^{(k)}))\right]y^{(k)}\\&+t_k(x^{(k)}-x^{(0)})=-(1-t_k)\big[\nabla f(x^{(k)})\lambda^{(k)}\\&+\sum_{\gamma\notin I(\bar{x})}(z_\gamma^{(k)}\nabla h_\gamma(x^{(k)})+t_k(z_r^{(k)})^2\alpha_r(x^{(k)}))\big]\end{aligned}\tag{3.17}$$

When $k\to\infty$, that

$$\sum_{\gamma\in I(\bar{x})}\lim_{k\to\infty}(1-t_k)\left[z_\gamma^{(k)}\nabla h_\gamma(\bar{x})+(z_r^{(k)})^2\alpha_r(\bar{x})\right]+\beta(\bar{x})\bar{y}+\bar{x}-x^{(0)}=0.\tag{3.18}$$

By the assumptions (H2) and (3.18) we have

$$\lim_{k\to\infty}(1-t_k)z_\gamma^{(k)}=0,\ \lim_{k\to\infty}(1-t_k)(z_\gamma^{(k)})^2=\bar{\rho}_r, r\in I(x),\tag{3.19}$$

where $\bar{\rho}_r\geq 0$. Therefore, from (3.18) and (3.19) we get

$$\bar{x}+\sum_{r\in I(x)}\bar{\rho}_r\alpha_r(\bar{x})+\beta(\bar{x})\bar{y}=x^{(0)}$$

which contradicts with the conditions(H3), thus $\|z^{(k)}\|\to\infty$ is impossible.

From (1), (2), (3)and(4)we conclude that $\Gamma_{w^{(0)}}$ is bounded.

Theorem 3.1 (Convergence of the method). If conditions of Lemma 3.1 are satisfied, then (1.1) has at least one solution. For almost all $w^{(0)}\in\Omega\times\Lambda^{++}\times R^m\times R_{++}^{l+p}\times\{0\}$, the zero-point set $H_{w^{(0)}}^{-1}(0)$ of homotopy mapping (3.1) contains a smooth curve $\Gamma_{w^{(0)}}$, which starts from $(w^{(0)},1)$. As $t\to 0$, the limit set $T\subset\bar{\Omega}\times\Lambda^{+}\times R^m\times R_{+}^{l+p}\times R\times\{0\}$ of $\Gamma_{w^{(0)}}$ is nonempty, and every point in T is a solution of (1.1).

Specifically, if the length of $\Gamma_{w^{(0)}}$ is finite and $(w^*,0)$ is the end point of $\Gamma_{w^{(0)}}$, then w^* is a solution of (1.1).

Proof. By Lemma 3.1, 0 is a regular value of $H_{w^{(0)}}$ for almost all $w^{(0)}\in\Omega\times\Lambda^{++}\times R^m\times R_{++}^{l+p}\times\{0\}$, and $H_{w^{(0)}}^{-1}(0)$ contains a smooth curve $\Gamma_{w^{(0)}}$ starting from $(w^{(0)},1)$.

By the classification theorem of one-dimensional smooth manifold, $\Gamma_{w^{(0)}}$ is diffeomorphic to a unit circle or the unit interval (0, 1]. Noticing that

$$\left.\frac{\partial H_{w^{(0)}}(w,t)}{\partial w}\right|_{\substack{w=w^{(0)}\\t=1}}=\begin{pmatrix}E_n&0&\beta(x^{(0)})&0&0&0\\0&E_p&0&0&0&-e\\z^{(0)}\nabla^T h(x^{(0)})&0&0&Z&0&0\\\nabla^T g(x^{(0)})&0&0&0&0&0\\0&\theta&0&0&\Lambda&0\\0&-e^T&0&0&0&0\end{pmatrix}$$

is nonsingular, we know that $\Gamma_{w^{(0)}}$ is not diffeomorphic to a unit circle. That is, $\Gamma_{w^{(0)}}$ is diffeomorphic to (0, 1].

Let $(\bar{w},\bar{t})$ be a limit point of $\Gamma_{w^{(0)}}$ as $t\to 0$. Only the following three cases are possible:

(i) $(\bar{w},\bar{t})\in\Omega\times\Lambda^{++}\times R^m\times R_{++}^{l+p}\times R\times\{1\}$;

(ii) $(\bar{w},\bar{t})\in\partial(\Omega\times\Lambda^{+}\times R^m\times R_{+}^{l+p})\times R\times(0,1]$;

(iii) $(\bar{w},\bar{t})\in\bar{\Omega}\times\Lambda^{+}\times R^m\times R_{+}^{l+p}\times R\times\{0\}$.

Because the equation $H_{w^{(0)}}(w,1)=0$ has only one solution $(w^{(0)},1)\in\Omega\times\Lambda^{++}\times R^m\times R_{++}^{l+p}\times R\times\{1\}$, the case (i) is impossible. In case (ii), there must exist a sequence of $(w^{(k)},t_k)\in\Gamma_{w^{(0)}}$ such that $h_\gamma(x^{(k)})\to 0$ for some $1\leq\gamma\leq l$. From the third equality of (3.1), we have $\|z^{(k)}\|\to\infty$, which contradicts Lemma 3.2.

As a conclusion, (iii) is the only possible case, and hence, $\bar{w}$ is a solution of (1.1).

Remark3.1. By Theorem 3.1, for almost all $w^{(0)}\in\Omega\times\Lambda^{++}\times R^m\times R_{++}^{l+p}\times R$, the homotopy (3.1) generates a smooth curve $\Gamma_{w^{(0)}}$. We call $\Gamma_{w^{(0)}}$ as the homotopy path. Tracing numerically $\Gamma_{w^{(0)}}$ from $(w^{(0)},1)$ until $t\to 0$, one can find a solution of (1.1). Let s be the arclength of $\Gamma_{w^{(0)}}$. We can parameterize $\Gamma_{w^{(0)}}$ with respect to s. That is, there exist continuously differentiable functions $w(s),t(s)$, such that

$$H_{w^{(0)}}(w(s),t(s))=0,\tag{3.20}$$

$$t(0)=1,\ w(0)=w^{(0)}.\tag{3.21}$$

Differentiating (3.20), we obtain the following theorem.

Theorem 3.2 The homotopy path $\Gamma_{w^{(0)}}$ is deter-mined by the following initial value problem to the ordinary differential equation

$$\frac{\partial H(w(s),w^{(0)},t(s))}{\partial(w,t)}\begin{pmatrix}\dot{w}(s)\\ \dot{t}(s)\end{pmatrix}=0 \tag{3.22}$$
$$\left\|\dot{w}(s),\dot{t}(s)\right\|=1,\ w(0)=w^{(0)},\quad t(0)=1.$$

And the w component of the solution point of (3.20), for $t(s^*)=0$, is the solution of (1.1).

IV. Tracing The Homotopy path

Use In this section, we discuss how to trace numerically the homotopy path $\Gamma_{\omega^{(0)}}$. A standard procedure is the predictor-corrector method [13], which uses an explicit difference scheme for solving numerically and a simple numerical example is given.

Algorithm 4.1 (MOP)'s Euler-Newton method).

Step 0: Give an initial point

$$(w^{(0)},1)\in\Omega\times\Lambda^{++}\times R^m\times R^{l+p}_{++}\times R\times\{1\}.$$

An initial step-length $d_0>0$ and three small positive numbers ε_1, ε_2, ε_3. let $k:=0$

Step 1: Compute the direction $\eta^{(k)}$ of predictor step:

(a) Compute a unit tangent vector $\zeta^{(k)}\in R^{(n+m+l+2p+2)}$ of $\Gamma_{\omega^{(0)}}$ at

(b) Determine the direction $\eta^{(k)}$ of the predictor step.

If the sign of the determinant $\begin{vmatrix}DH_{\omega^{(0)}}(\omega^{(k)},t_k)\\ \zeta^{(k)^T}\end{vmatrix}$ is $(-1)^{ml+q}$, then $\eta^{(k)}=\zeta^{(k)}$

If the sign of the determinant $\begin{vmatrix}DH_{\omega^{(0)}}(\omega^{(k)},t_k)\\ \zeta^{(k)^T}\end{vmatrix}$ is $(-1)^{ml+q+1}$, then

$$\eta^{(k)}=-\zeta^{(k)}\ ,\ q=\operatorname{sign}\left|-\nabla^T g(x^0)\beta(x^0)\right|.$$

Step 2: Compute a corrector point ($\omega^{(k+1)},t_{k+1}$).

$$(\bar{\omega}^{(k)},\bar{t}_k)=(\omega^{(k)},t_k)+d_k\eta^{(k)},$$
$$(\omega^{(k+1)},t_{k+1})=(\bar{\omega}^{(k)},\bar{t}_k)-DH_{\omega^{(0)}}(\bar{\omega}^{(k)},\bar{t}_k)^+H(\bar{\omega}^{(k)},\bar{t}_k),$$

where

$$DH_{\omega^{(0)}}(\omega,t)^+=DH_{\omega^{(0)}}(\omega,t)^T(DH_{\omega^{(0)}}(\omega,t)DH_{\omega^{(0)}}(\omega,t)^T)^{-1}$$

is the Moore–Penrose inverse of $DH_{\omega^{(0)}}(\omega,t)$.

If $\left\|H_{\omega^{(0)}}(\omega^{(k+1)},t_{k+1})\right\|\le\varepsilon_1$, let $d_{k+1}=\min\{d_0,2d_k\}$, go to Step 3: If $\left\|H_{\omega^{(0)}}(\omega^{(k+1)},t_{k+1})\right\|\in(\varepsilon_1,\varepsilon_2)$, let $d_{k+1}=d_k$, go to Step 3.

If $\left\|H_{\omega^{(0)}}(\omega^{(k+1)},t_{k+1})\right\|\ge\varepsilon_2$, let $d_{k+1}=\max\left\{2^{-25}d_0,\frac{1}{2}d_k\right\}$, go to Step 2.

Step 3: If $w^{(k+1)}\in\Omega\times\Lambda^+\times R^m\times R^{l+p}_+\times R$ and $t_{k+1}>\varepsilon_3$, let $k=k+1$, go to Step 1.

If $w^{(k+1)}\in\Omega\times\Lambda^+\times R^m\times R^{l+p}_+\times R$ and $t_{k+1}<-\varepsilon_3$, let $d_k:=d_k\frac{t_k}{t_k-t_{k+1}}$, go to Step 2 and re-compute $(\omega^{(k+1)},t_{k+1})$ for the initial point $(\omega^{(k)},t_k)$.

If $w^{(k+1)}\notin\Omega\times\Lambda^+\times R^m\times R^{l+p}_+\times R$, let $d_k:=\frac{d_k}{2}\frac{t_k}{t_k-t_{k+1}}$, go to Step 2 and re-compute $(\omega^{(k+1)},t_{k+1})$ for the initial point $(\omega^{(k)},t_k)$.

If $w^{(k+1)}\in\Omega\times\Lambda^+\times R^m\times R^{l+p}_+\times R$ and $t_{k+1}\le\varepsilon_3$, then stop.

Remark4.1 In Algorithm 4.1, the arclength parameter s is not computed explicitly. The tangent vector at a point on $\Gamma_{\omega^{(0)}}$ has two opposite directions, one (the positive direction) makes s increase, and another (the negative direction) makes s decrease, The negative direction will lead us back to the initial point, so we must go along the positive direction. The criterion in Step 1 (b) of Algorithm 4.1 that determines the positive direction is based on a basic theory of homotopy method [14], that is, the positive direction η at any point (ω,μ) on $\Gamma_{\omega^{(0)}}$ keeps the sign of the determinant $\begin{vmatrix}DH_{\omega^{(0)}}(\omega,t)\\ \eta^T\end{vmatrix}$ invariant. We have the following proposition.

Proposition 4.1 If $\Gamma_{\omega^{(0)}}$ is a smooth curve of $H_{\omega^{(0)}}{}^{-1}(0)$. Then the direction $\eta^{(0)}$ of the predicted step at the initial point $(\omega^{(0)},1)$ satisfies

$$\operatorname{sign}\begin{vmatrix}DH_{\omega^{(0)}}(\omega^{(0)},1)\\ \eta^{(0)^T}\end{vmatrix}=(-1)^{ml+q}\,..$$

Proof: From

$$DH_{\omega^{(0)}}(\omega^{(0)},1)=$$
$$\begin{bmatrix} E_n & 0 & \beta(x^{(0)}) & 0 & 0 & 0 & a\\ 0 & E_p & 0 & 0 & 0 & -e & b\\ z^{(0)}\nabla^T h(x^{(0)}) & 0 & 0 & Z & 0 & 0 & -z^{(0)}\circ h(x^{(0)})\\ \nabla^T g(x^{(0)}) & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \theta & 0 & 0 & \Lambda & 0 & -\xi^{(0)}\circ\lambda^{(0)}\\ 0 & -e^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$=(M_1\ \ M_2)$$

where

$$a=-\lambda^{(0)}\nabla f(x^{(0)})-z^{(0)}\nabla h(x^{(0)})-\alpha(x^{(0)})(z^{(0)})^2,$$
$$\beta=-f(x^{(0)})-\xi^{(0)},$$
$$M_1\in R^{(n+l+m+2p+1)\times(n+l+m+2p+1)},M_2\in R^{(n+l+m+2p+1)\times 1},$$

M_1 is nonsingular and $M_2\neq 0$. the unit tangent vector $\zeta^{(0)}$ of $\Gamma_{\omega^{(0)}}$ at $(\omega^{(0)},1)$ satisfies

$$(M_1 \quad M_2)\begin{pmatrix} \zeta_1^{(0)} \\ \zeta_2^{(0)} \end{pmatrix} = 0,$$

$$\zeta_1^{(0)} \in R^{(n+l+m+2p+1)}, \zeta_2^{(0)} \in R.$$

Define $\zeta^{(0)} \equiv (\zeta_1^{(0)}, \zeta_2^{(0)})^T$, by a simple computation, we have

$$\zeta_1^{(0)} = -M_1^{-1} M_2 \zeta_2^{(0)},$$

$$|M_1| = -e^T e \prod_{i=1}^{p} \lambda_i \prod_{j=1}^{l} z_j (-1)^{ml} \left| -\nabla^T g(x^0) \beta(x^0) \right|.$$

Hence

$$\begin{vmatrix} DH_{\omega^{(0)}}(\omega^{(0)},1) \\ \zeta^{(0)^T} \end{vmatrix} = \begin{vmatrix} M_1 & M_2 \\ \zeta_1^{(0)^T} & \zeta_2^{(0)^T} \end{vmatrix} = \begin{vmatrix} M_1 & M_2 \\ -M_2^T M_1^{-T} & 1 \end{vmatrix} \zeta_2^{(0)}$$

$$= \begin{vmatrix} M_1 & M_2 \\ 0 & 1 + M_2^T M_1^{-T} M_1^{-1} M_2 \end{vmatrix} \zeta_2^0$$

$$= |M_1| (1 + M_2^T M_1^{-T} M_1^{-1} M_2) \ \ \zeta_2^0$$

Because $\lambda_i \geq 0, (i \in P), z_j \geq 0, (j \in L)$ and by the definition of the direction of the predictor step, $\eta_2^{(0)} < 0$ and

$$(1 + M_2^T M_1^{-T} M_1^{-1} M_2) > 0$$

$$\operatorname{sign} |M_1| = (-1)^{ml+q+1}$$

$$q = \operatorname{sign} \left| -\nabla^T g(x^0) \beta(x^0) \right|$$

So

$$\operatorname{sign} \begin{vmatrix} DH_{\omega^{(0)}}(\omega^{(0)},1) \\ \eta^{(0)^T} \end{vmatrix} = (-1)^{ml+q}.$$

In the following, we have tested the homotopy method by a simple numerical simulation.

Example3.1 f, h and g are defined as in Problem(P1),and we set $n = 2, p = 2, l = 2, m = 1$,

$$\min f_1(x) = x_1^2$$

$$f_2(x) = x_2^2$$

$$\text{s.t} \begin{cases} h_1(x) = -x_2 - 6 \leq 0 \\ h_2(x) = x_2 - 6 \leq 0 \\ g(x) = x_1 - x_2^2 - 3 = 0 \end{cases}$$

It is easy to see that the assumption (A_3) of reference[9]is not satisfied in $\overline{\Omega}$. Let $\alpha_r(x) = \nabla h_r(x), (r = 1,2)$ and $\beta(x) = (10,0)^T$. It is easily verified that $\overline{\Omega}$ satisfies the assumption $(H_1) - (H_4)$.

Let

$$(x_1^{(0)}, x_2^{(0)}) = (7.0, 2.0), (\lambda_1^{(0)}, \lambda_2^{(0)}) = (0.5, 0.5)$$

$$(z_1^{(0)}, z_2^{(0)}) = (1.0, 1.0),$$

$$y^{(0)} = 0,0 \ h^{(0)} = 0.0, (\xi_1^{(0)}, \xi_2^{(0)}) = (2.0, 1.0),$$

The numerical results of $x^*, \lambda^*, z^*, y^*, h^*, \xi^*$ are listed in Table I.

TABLE I. NUMERICAL SIMULATION RESULTS OF EXAMPLE 4.1

x^*	λ^*	y^*
(3.0000, 0.0001)	(0.0000,1.0000)	0.0003
z^*	ξ^*	h^*
（0.0001，0.0001）	(9.0000, 0.0000)	0.0000

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