Application of Hurst Exponent (H) and the R/S Analysis in the Classification of FOREX Securities

Milton S. Raimundo and Jun Okamoto Jr

Abstract—**This paper presents the relationship between the Hurst Exponent (H) and the Rescaled Range Analysis (R/S) in the classification of Foreign Exchange Market (FOREX) time series by the supposition of the existence of a Fractal Market in an alternative to the traditional theory of Capital Markets. In such a way, the Hurst Exponent is a metric capable of providing information on correlation and persistence in a time series. Many systems can be described by self-similar fractals as Fractional Brownian Motion, which are well characterized by this statistic.**

Index Terms—**Hurst exponent, R/S analysis, fractal analysis, financial time series, fractional.**

I. Introduction

The necessity to anticipate and identify changes in events points to a new direction in line with the analysis of the fluctuations of prices of financial assets. This new direction leads us to argue about new alternatives in Finance Theory and Capital Markets. In the spirit of this contention the theory of fractals arises by innovating the argumentation [1].

Empirical studies, especially in Hydrology and Climatology, decade of 1950, reveal the presence of Long Memory (LM) in temporal and spatial data. These series present persistence in the sample autocorrelations, that is, significant dependence between observations separated by a long interval of time [2].

Harold Edwin Hurst [3] has spent much of his academic life studying water storage issues. He invented a new statistical method, Hurst Exponent (H) that is applied in several areas including Fractals and Chaos Theory, Long Memory Processes and Spectral Analysis. It provides concrete information on correlation and persistence, which makes H an excellent index for studying complex processes such as the financial time series. The values of the Hurst Exponent of time series are estimated using the Rescaled Range Analysis (R/S) method.

The value of this exponent varies from 0 to 1. The more distinct from 0.5, the longer the long-term memory. Therefore, processes with long memory are processes that have $H > 0.5$, persistent processes, or $H < 0.5$, anti-persistent processes. For $H = 0.5$ the signal or process is random or Brownian Motion [4].

This paper aims to present the relationship between the Hurst Exponent and the R/S Analysis in the prediction of financial time series by the hypothesis of the existence of a Fractal Market in an alternative to the traditional theory of Capital Markets. The method used for the estimated calculation of the Hurst Exponent (H), using the R/S Analysis, follows the methodology developed by Mandelbrot and Wallis [4], based on the works of Hurst [5], discussed by Peters [6] and presented by Couillard [7].

Some academic research initiatives have been developed and performed in the field of econophysics to examine the properties and phenomena of random walk and presence of long memory in financial time series.

In his paper, Qian [8] analyzed the Hurst exponent for Dow-Jones index. He found that the periods with large Hurst exponents could be predicted more accurately than those with H values close to random series, which suggested that stock markets were not totally random in all periods.

Corazza [9] studied the returns in some FX Markets and he discovered that the values of Hurst Exponent were statistically different from 0.5 in most of the samples. He also found that the Hurst exponent is not fixed but it changes dynamically over time, concluding that FX Markets follow a fractional Brownian motion.

Cajueiro [10] measured the long-term dependence and efficiency in emergence markets of stock indexes. He suggested that the long-range dependence measures are more significant with Asian countries than for Latin American countries.

Others, like Cornelis [11], Singh [12] and Kyaw [13], also studied the degree of long-term dependence and the fractional Brownian motion in several return curves by the application of the Hurst Exponent.

Hence, despite the good results found in several studies, the challenge in the prediction of financial time series by the hypothesis of the existence of the Fractal Market is still open, either for forecasting financial time series or for forecasting trends for financial indicators.

This paper is organized as following: Section 2, Fractal Time Series, presents a review of fractal theory relating it to time series. Section 3, Hypothesis of the existence of a Fractal Market, compares the Fractal Market and the traditional Theory of Capital Markets. The Section 4, Hurst Exponent and the R/S Analysis, exposes the theoretical bases of the Hurst processes and the development of the R/S Analysis. Section 5, Methodology, exposes the method used for the estimated calculation of the Hurst Exponent (H), using the R/S Analysis. Section 6, Experimental Results, presents the R/S Analysis evaluation in a use case. Finally, Section 7, Conclusion, presents the general analysis and the contributions of this paper, as well as suggestions for future papers.

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II. Fractal Series

It can be seen that many of the spatial structures in nature are the result of the combination of a considerable number of identical components, implying the existence of the principle of having similar self-structures called fractal [14].

The concept of fractals relates to attempts to measure the size of the targets for which traditional definitions based on Euclidean geometry do not work well. A fractal, usually of fractional size, is a mathematical entity that can present itself as a spatial pattern or even a time series, and can be divided into parts, where each of these parts is similar to the original object [15].

A. *Fractal Geometry*

Fractal is defined by a geometric figure, of fragmented appearance, which can be subdivided indefinitely into parts, where the parts are, in some way, reduced copies of the whole. There is a geometric fractal when the whole is a perfect magnification of a part. The concept of geometric fractal was created in the early twentieth century to show that there were mathematical elements different from what the classical geometry of Euclid (300 BC) said [16].

B. *Deterministic Fractals*

Exact self-similarity is understood to be the invariance of the structure after an isotropic transformation, that is, that which occurs with the same intensity in all directions. In this matter, by taking an object S, as a set of points $R = \{x_1, x_2, x_3 ...\}$ and applying a factor of scale b in a similar auto transformation, it changes the coordinates of the points for $bR = \{bx_1, bx_2, bx_3, ...\}$. Then the set S composed by the points of coordinates R, is self-similar if it is invariant after performing the transformation [17].

As an example, a fractal called the Sierpinski Triangle shown in Fig. 1(a). Its basic construction begins with an equilateral triangle, fully filled. Considering initially the midpoints of the three sides that, together with the vertices of the original triangle, form four new congruent triangles.

Subsequently, the central triangle is subtracted, thereby removing the first stage of the basic building process. This subtraction results in three congruent triangles whose sides measure half the side of the original triangle. The procedure described above is repeated for each of these three triangles.

In this way, starting with a single triangle, a sequence of 3, 9, 27, 81 triangles are generated, corresponding respectively to levels 1, 2, 3 and 4, presented in Fig. 1(b) [17].

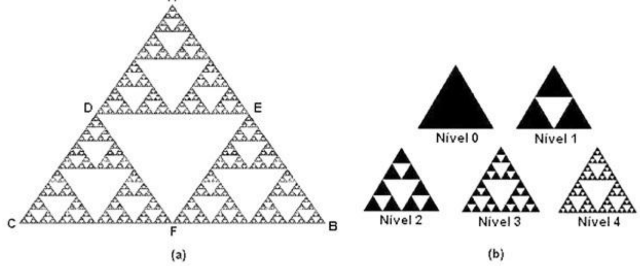

Fig. 1. (a) Sierpinski Triangle and (b) its first five constructions [17].

This formation law, successively applied, causes the triangular structure to consist of gradually smaller triangles that are perfect copies of the initial shape of the figure. Hence, by applying a zoom to any part has something similar to the figure as a whole as exposed in Fig. 1(a). In the extreme of the infinite applications of this process is obtained an exact similar auto-fractal figure [17].

C. *Fractal Dimension*

Assis [17] also emphasizes that by the classical Euclidean concept a coordinate, length, describes a line, two coordinates, length and width, describe a plane and three coordinates, length, height and width, and describe a volume. By this prism, a point has dimension zero.

Associated with the perpendicular axes, the Euclidean dimension specifies in one, two or three dimensions some point respectively belonging to a line, area or volume. It may be noted that Euclidean dimensions are always integers.

Considering the Koch curve, exhibited in Fig. 2, which construction is done by using a line segment, which is divided into three equal segments. From there, it replaces the third median part with an equilateral triangle removing the base. The iterative procedure is the application of the rule to each of the line segments that resulted from the previous iteration [17].

Observing each step of the iterative process, it is noted that, from one level to the next, three segments are replaced by four of equal length, so the total length is multiplied by ¾ in the correlation of successive levels. It also be seen that the limit of a geometric succession of ratio ¾ is infinite, determining that the final figure will have an infinite length. Mandelbrot also had observed that and denominated this limit.

Thus, in the nth level, the length of the Kock curve will be given by:

$$L_n = L_{n-1} + (L/3) = (4/3)^n \quad (1)$$

Fig. 2 represents the first four levels for the construction of the Kock curve and their respective lengths.

Comparing this type of curve, which has infinite detailing, with a conventional line, this curve occupies more space, and hence has a fractal dimension greater than 1 : 0, even so it does not occupy the space of a band that contains it, in this case dimension 2 : 0. Because of that, this concept of fractal dimension, *D*, is closely related to the structure of occupation of the space of the figure.

According to Falconer [18], the topological dimension D_t can be defined iteratively by defining the topological dimension of a point as zero. The topological dimension of other objects is a result of the value of D_t of the element which makes it disconnected plus 1. For a curve, one point is sufficient to make it disconnected, so that the corresponding value of D_t is 0 + 1 = 1. Continuing with this logic, for a plane, a curve makes it disconnected, thus having 1 + 1 = 2. Finally, for a volume, a surface makes it disconnected, 2 + 1 = 3.

Also, according to Falconer [18], the immersion dimension, D_i, represents the dimension in which the object is immersed.

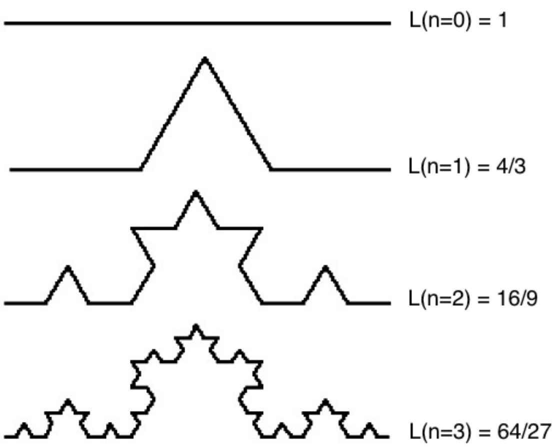

Fig. 2. Kock curve showing the first 4 levels and their lengths [17].

For instance, the letters of this paper are taken. These have topological dimension 1, however they are immersed in the space of the paper sheet, which suggests that their immersion dimension is 2.

The fractal dimension of a set, begins with the definition of metric space R^n. In the line a range is defined as a line segment, whereas in R^n the interval is defined as a sphere of radius γ, centered at x_i. The sphere is represented by $B(x_i;\gamma)=\{y_i \in R^n \mid d(x_i;y_i)<\gamma\}$ where $d(x_i;\ y_i)$ is the measure of the space R^n.

In accordance with Mandelbrot [19], a given set A constitutes a fractal if, in A, $D_i > D > D_t$, where D is the fractal dimension, D_i is the immersion dimension, and D_t is the topological dimension of the set A.

III. Fractal Market

The Efficient Markets Hypothesis has some exceptions to the assumption of normality [20]. Osborne [21] had found an anomaly when attempting to plot a graph of the probability density function of the distribution of returns. Osborne also observed that the distribution had higher kurtosis than the normal, leptokurtic, or heavy-tailed distribution, although at that time he had not madden attention to this fact.

Turner and Weigel apud Peters [6] conducted a study on the behavior of the volatility of the daily returns of the S&P Index for the period between 1928 and 1990, finding results similar to those found by Fama [22] and Sharpe [23].

These studies show evidence that returns on securities in the capital markets are not normally distributed. Hence, if returns are not normally distributed, much of the statistical analysis that of correlation coefficients, becomes highly compromised and may lead to erroneous results, suggesting also that the idea of the occurrence of random walk in stock prices is wrong.

The Fractal Theory began to impose itself as a counterpoint to the hypotheses created by the more conservative financial theory, for offering a more realistic perspective on the functioning of the Financial Market.

Called self-similar, the fractals have less elaborate structures, whose scale changes are manifested proportionally in every structure. The fractals most similar to the behavior of prices are the self-related ones, since they change of scales in diverse directions. There are also multi-fractals, those in which scales vary in several different ways.

When analyzing fractal geometry figures, it possible to notice a similarity between the parts with the whole, independent of the scale in which the object is observed. This property describes fractal behavior governed by the Power Law.

The classical Power Law, initially discovered by Vilfredo Pareto in studies on income distribution, allows us to visualize changes in patterns of behavior that are repeated in numerous size scales of the figure. This law is characterized by the probability of measuring some quantity that varies exponentially with that quantity. But was through Mandelbrot that in 1961 identified the Power Law in the series of prices of assets in the Financial Market. Later Mandelbrot published an article [24], aided by computational processing, where he identified patterns in the price series governed by a similarity defined by the Power Law and also described that the behavior of asset prices was distant from a normal distribution.

Stable distributions are characterized by four parameters: α, β, γ and δ. The parameter α refers to the characteristic exponent, ranging from 0 and 2; $\alpha = 2$ corresponds to a Gaussian distribution. The parameter β means asymmetry and varies between -1 and 1. The parameters γ and δ are corresponding to the scaling and location parameters respectively.

The fractal dimension allows the identification of the existence of Power Law, is given by $P = n^D$, where P are similar auto parts, n is magnification factor or Power Law governing the geometry, and D is the fractal dimension.

Solving and evidencing in D, results in:

$$D=\left(\log(P)\big/\log(n)\right) \quad (2)$$

In this way, Mandelbrot calculated the slope of the line, $\alpha = 1.7$; smaller than the α of the normal distribution, $\alpha = 2$, indicating a Power Law.

Consequently, Mandelbrot developed two mathematical tools of fractal analysis in time series: the α distribution and the Hurst Exponent (H). Therefore, α is a measure of asset risk value, the smaller the value of the α, the heavier the tail behavior in the distribution, and the Hurst Exponent (H) reflecting the existence of persistence, a specific Power Law, driving the asset prices behavior.

Mandelbrot [24] also developed a new statistical measure of nonparametric analysis called the Rescaled Range Analysis (R/S). By this new metric $H = 1/\alpha$. In the hypothesis that $H = ½$ means a Brownian motion, or rather having a normal distribution. Hence, if $H = ½ \Leftrightarrow \alpha = 2$ normal curves.

A. Analysis of the Evidence of Fractals in the Financial Series

Peters, [25] using the Hurst R/S technique, evaluated the persistence of memory in series of monthly returns of stocks of the S&P500 Index, American T-Bonds, and the relative return between the two series between the years of 1950 and 1988, in a total of 463 monthly observations.

Using the Hurst Exponent, or the power factor, H = log

(*R/S*) / *N*, where *R/S* is the division of the amplitude between the highest and the lowest occurrence recorded (*R*), by the standard deviation found in the series (*S*) and *N* is the number of observations, Peters obtained the results: $H = 0.61$ for the Index, $H = 0.64$ for the T-Bonds and $H = 0.66$ for the relationship between Index and T-Bonds. Peters evidenced a behavior of memory persistence of the market, characteristic of Brownian fractal behavior, but not significant to the point of the projection of the results beyond a short period since the random factor was much more present. Hence, he can conclude that the results pointed out that the model, called pure random walk, and did not apply to capital markets, as the efficient markets hypothesized.

Müller [26] found fractals characteristics in time series of exchange rates, also identifying the scaling factor, the Power Law, followed by prices, by analyzing the time intervals of daily and intra-daily samples between the years from June 1973 to June 1993, from a few minutes to a year. Müller can also verify that the changes in the price behavior were more similar to the fractal model, when compared them with GARCH processes. Besides that, volatility was positively correlated with the market activity and the transaction volume, in an indication that the market was also heterogeneous.

Corazza [27] worked with commodity futures prices on the Chicago Stock Exchange between 1981 and 1991. He applied four tests, including the Hurst technique. The result found indicated that the historical series shared the fractional properties.

Several other authors such as Larrain [28], Lo [29], Barkoulas [30], Richards [31], Panas [32] and more recently Di Matteo [33], among others, reached the conclusion that all global financial markets presented scale symmetry, with fractal characteristics, as well as the methodologies usually used to control risk, based on the standard deviation, were not efficient and able to provide a good classification.

IV. HURST EXPONENT AND THE R/S ANALYSIS

Suggested by the English engineer and hydrologist Harold Hurst in 1951, the H index estimate, part of the Rescaled Range Analysis (R/S_t). Hurst was responsible for building dams along the Nile River, when he observed the accumulation of values above and below the mean while tabulating the river level over the years. Inspired by the random process definition suggested by Einstein, Hurst ran some tests to check if the accumulation of above and below average values was random.

Let a sequence of values ε_t, $t \geq 1$, be independent and identically distributed (*i.i.d.*), with mean μ_ε and variance σ_t^2.

The sequence $W_t = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_t$ is defined as a random walk and also:

$$E(W_t) = E(\varepsilon_1) + E(\varepsilon_2) + \ldots + E(\varepsilon_t) = t\mu_\varepsilon \quad (3)$$

$$Var(W_t) = Var(\varepsilon_1) + Var(\varepsilon_2) + \ldots + Var(\varepsilon_t) = t\sigma_\varepsilon^2 \quad (4)$$

where $E(\varepsilon)$ is the Expectation Value of ε and $Var(\varepsilon)$ is the Variance of ε.

If the series of the level of the Nile River were set as a random walk, then the standard deviation value would be equal to $\sigma_\varepsilon t^{½}$. The Rescaled Range Analysis (R/S_t) tests the null hypothesis of the exponent of the variable t being equal to ½. Calling this exponent as H, homage of Mandelbrot to Hurst and the mathematician Ludwing Otto Hölder [34], it is possible to define the test by the null hypothesis H_0: $H = ½$ and by the alternative hypothesis H_0: $H \neq ½$.

Mandelbrot [35] has also developed a method to estimate the H parameter. By this method rejects the null hypothesis because $H > ½$, it is the case of persistence, rejects the null hypothesis because $H < ½$, it is the case of anti-persistence.

It is possible to follow the method developed by Mandelbrot proceeding the presentation of Couillard [7], also discussed by Peters [6], according to:

Let be a time series with *N* observations for equal intervals of time. Dividing the observations into *M* sub-periods with the same number of observations *t*, such as $M \times t = N$, I_m is defined for $m = 1, 2, \ldots, M$, such as each of the M sub-periods and $N_{k,m}$, $k = 1, 2, \ldots, t$, as each element of a given sub-period. By this defines the mean μ_{I_m} and the standard deviation σ_{I_m} of each sub-period. The notation adopted by this paper follows the same notation presented by Couillard [7] and used by Peters [6]. In this case $\sigma_{I_m} = S_{I_m}$ to respect the notation R/S for the Rescaled Range formula.

$$\mu_{I_m} = \left(1/t\right)\sum_{k=1}^{n} N_{k,m} \quad (5)$$

$$S_{I_m} = \sqrt{\left(1/t\right)\sum_{k=1}^{n}\left(N_{k,m} - \mu_{I_m}\right)^2} \quad (6)$$

Having the values of the means it is conceivable to reconstruct the original series and obtain a series with *M* sequences I_m composed, each, by *t* deviations accumulated with respect to μ_{I_m}. These accumulated deviations are defined as follows:

$$X_{k,m} = \sum_{k=1}^{n}\left(N_{k,m} - \mu_{I_m}\right) \quad (7)$$

where the amplitude of the mean deviations accumulated in each sequence I_m is defined by:

$$R_{I_m} = \max\left(X_{k,m}\right) - \min\left(X_{k,m}\right) \quad (8)$$

The series with M values R_{I_m} is normalized by dividing these amplitude values by their corresponding standard deviations S_{I_m}. The mean of these standard values maintains the relationship between H and *t*. Therefore, the statistic is defined as:

$$R/S_t = \left\lfloor \left(1/M\right)\sum_{m=1}^{M}\left(R_{I_m}/S_{I_m}\right)\right\rfloor = ct^H \quad (9)$$

where c is a constant.

To obtain the value of H, and test it, computes a series of statistics R/S_t for different values of t, linearize the equality $R/S_t = ct^H$ and, with that, estimate the value of H. To linearize the equality $R/S_t = ct^H$, simply apply the logarithm:

$$\log\left(R/\mathrm{S}_t = \log(c) + H\log(t)\right) \quad (10)$$

The value of H can be estimated by simple linear regression. Since R_{I_m} is always greater than or equal to zero and S_{I_m} is always greater than zero, the value of H will have a lower limit close to zero, depending on the value of c. As $R_{I_m} = S_{I_m}$ are sums of t normalized values, their maximum value tends to t, so the maximum value of H tends to 1, depending on c.

The question in performing this test is to define the size of the sub-periods I_m so as to preserve for each value of R/S_t a number as close as possible to the variables. Also, the values of t should preferably be entire divisors of N, or integers as close as possible to some divisor of the series size, and consequently that the amount of deleted data may be minimal.

Hurst [5] in his original test, needed the hypothesis that the values were normally distributed. By the test suggested by Mandelbrot [35], a test t, potentially the errors of the statistical model are associated with equation (13) as *i.i.d.* and normal. There is a discussion about the validity of these tests. Lo [29] and MacKinlay [36] attest that the tests may lead to the conclusion that there is long-term memory, persistence or anti-persistence, when in fact there are only short-term autocorrelation. The two tests, Hurst [5] and Mandelbrot [35], would not be robust in the presence of correlations between nearby variables.

However, there is evidence that the suggested correction has a bias. As observed by Teverovsky [37], Mandelbrot and Hudson [34], in simulated environments, had noticed the propensity to accept the null hypothesis H_0: $H = ½$ when it is false.

Couillard [7] suggested a specific t test for the H index. The controversy in applying a test for the H index by the way in which the finite sequences of values defined as empirically random walks are shown as values greater than ½ for the H index. Note that the values of H are distributed as a *t-Student* curve. This author proposes the test $t = (v - m) / d$, where v is the estimated value of H, m is the mean of the value of H if there is no long memory and d is the standard deviation of H if there is no long memory.

V. Methodology

The method used for the estimated calculation of the Hurst Exponent (H), using the R/S Analysis, follows the methodology developed by Mandelbrot and Wallis [4], based on the works of Hurst [5], discussed in Section 4. It is also included a first task to select the data of the elected time series for the usage of the R/S Analysis, scope of this work. So, the execution of the method presented here consists of the execution of the following tasks, the following tasks, highlighted in the presentation of Couillard [7], as well emphasized in Peters [6]:

- Selection of time series data;
- Calculation of the mean (E_m) and the standard deviation (S_m) of the sub-series ($Z_{i,m}$);
- Standardization of the subseries data ($Z_{i,m}$) by subtraction of the mean of the sub-series $X_{i,m} = Z_{i,m} - E_m$, for $i = 1, \ldots, N$;
- Generation of the cumulative time series $Y_{i,m} = \sum_{j=1}^{i} X_{j,m}$, for $i = 1, \ldots, N$;
- Calculation of the variation $\mathrm{Rm} = max\{Y_{1,m}, \ldots, Y_{n,m}\} - min\{Y_{1,m}, \ldots, Y_{n,m}\}$;
- Computation of the Statistics (R_m/S_m); and
- Estimation of the mean $(R/S)_n$ of the statistic (R_m/S_m) of all subsets of size n.

Observing that the statistic R/S asymptotically follows the relation $(R/S)_n \approx cn^H$, the value of the Hurst Exponent (H) can be calculated by a simple linear regression: $\log(R/S)_n = \log c + H \log n$.

If the process is a Brownian motion, H must be ½, when the process is persistent then H is greater than ½, and when anti-persistent H is less than ½. For a simple linear trend, $H = 1$ and for a white noise $H = 0$. Therefore, H must be between 0 and 1.

Beyond that, according to Couillard and Davison [7] most studies fail to find $H \neq ½$ because they do not provide a significance test. That way, the test suggested by de Couillard and Davison is adopted and performed using the *p–value* < 0.001.

VI. Experimental Results

A. Selection of the Time Series

For the experimental part, five financial time series were analyzed using the log-returns of Foreign Exchange Market (FOREX) [38] assets type.

The Foreign Exchange Market (FOREX, FX, or Currency Market) is a global market for decentralized currency trading [39]. In terms of trading volume, it is by far the largest market in the world [40]. The main participants in this market are the largest international banks and they are responsible for defining the relative values of the different currencies.

The following relationships between globalized currencies are used: AUD - Australian Dollar → JPY - Japanese Yen, CHF - Swiss Franc → JPY - Japanese Yen, EUR - Euro → JPY - Japanese Yen, GBP - British Pound → JPY - Japanese Yen and EUR - Euro → CHF - Swiss Franc.

According to Morettin [2] the price variation between the instants $t–1$ and t is given by $\Delta P_t = P_t - P_{t-1}$ and the relative price or the return of this asset is defined by $R_t = [(P_t - P_{t-1}) / (P_{t-1})] = [(\Delta P_t) / (P_{t-1})] = \{[(P_t) / (P_{t-1})] - 1\}$ or rather:

$$1 + R_t = (P_t)/(P_{t-1}) \quad (11)$$

Still according to Morettin [2], normally R_t is expressed in percent, being called the rate of return. Also denoted by $p_t = \log P_t$, where log is the e base logarithm, the log-return of financial assets is defined as:

$$r_t = \log\left[\left(P_t\right)/\left(P_{t-1}\right)\right] = \log\left(1+R_t\right) = p_t - p_{t-1} \quad (12)$$

The log-returns of the listed FOREX assets were calculated from the prices between 01/01/2003 and 12/30/2014 in intervals of 1 minute, obtained from the public knowledge base [38].

A framework is created, developed in Visual Studio .NET, Framework 4.6.2 [41], 64-bit, Windows Platform, for generation and selection of 3 new periodicities to execute the tests, with a quantity around 3000 values per series generated and selected, in intervals of 1 day, 1 hour and 15 minutes.

Thereafter, using data mining algorithms, the data from the time series are treated by eliminating irrelevant data from the set of results to be processed which includes correcting in the data and adjusting in the formatting of the data.

A function called of Price Tunneling, developed in the programming area of the Statistical Data Analysis *R* [42], version 3.1.2 of 10/31//2014, is used to check for price distortions in a given period of the series, by the percentage change ΔP_t of the current value of the price P_t in relation to the previous price P_{t-1}. Usually this percentage variation ΔP_t, adopted in the financial markets, is not more than 5% of the current price P_t both sides, up and down, over the previous price P_{t-1}.

In this way, the variation ΔP_t, respecting the price tunnel, is expressed as: $0.95 < \Delta P_t < 1.05$.

If the current price P_t is outside the price tunnel, the algorithm generates a random price P_t, from the previous price P_{t-1}, respecting the price tunnel boundaries.

B. Tests and Experiments

To prove that the markets are Hurst processes or biased random walks, the tasks were executed according to the method adopted, described above.

The historical series of prices chosen contain approximately 3000 values for the pairs of currency prices: AUD-JPY, CHFJPY, EUR-JPY, GBP-JPY and EUR-CHF, in intervals of 1 day, 1 hour and 15 minutes, generated by a framework developed in Visual Studio .NET, Framework 4.6.2 [41] 64-bit, Windows Platform, obtained from a public knowledge base [38], with prices varying between 01/01/2003 and 12/30/2014 in intervals of 1 minute.

These historical series of prices were treated by the function Price Tunneling to eliminate distortions in the prices of the series and later their log-returns were divided in 15 different values of *t*, periods or sub-series, to calculate the statistics R/S_t.

TABLE I: ESTIMATION VALUES OF THE HURST EXPONENT (H), USING THE R/S ANALYSIS, FOR THE RELATIONSHIP BETWEEN AUD-JPY CURRENCIES WITH 1 DAY INTERVALS.

t	*M*	R/S_t	$x = \log\{t\}$	$y = \log\{R/S_t\}$
3	1003	1.105242	1.098612	0.1000647
5	602	1.716025	1.609438	0.5400103
8	376	2.457615	2.079442	0.8991914
13	231	3.361801	2.564949	1.2124767
21	143	4.608474	3.044522	1.5278967
34	89	6.25723	3.526361	1.8337377
55	55	8.262083	4.007333	2.1116767
89	34	10.58514	4.488636	2.3594511
144	21	13.201705	4.969813	2.580346
233	13	18.464635	5.451038	2.9158573
377	8	22.790831	5.932245	3.1263583
610	5	27.968833	6.413459	3.3310908
987	3	29.541445	6.89467	3.3857942
1597	2	40.33256	7.375882	3.6971591

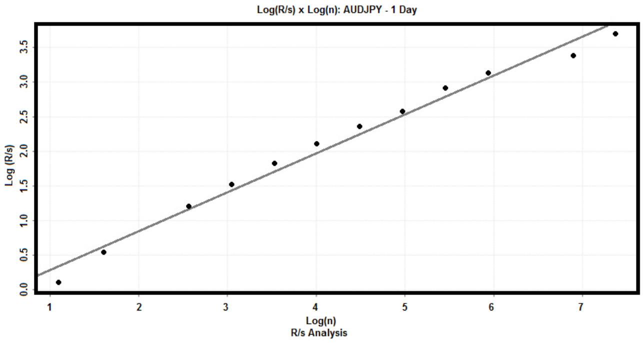

Fig. 3. Scatter plot with the values of $\log\{R/S_t\}$ in relation to the values of $\log\{t\}$, for the relationship between AUD-JPY currencies with 1-day intervals, and its trend line.

Details of the algorithm Price Tunneling can be verified in the previous section.

In order to obtain the mean (E_m), the standard deviation (S_m) and the generation of the sub-series ($Z_{i,m}$) three functions were used, developed in the programming area of the Statistical Data Analysis *R* [42], version 3.1.2 of 10/31/2014.

For the normalization of the sub-series data ($Z_{i,m}$), as well as the generation of the cumulative time series ($Y_{i,m}$), the *cumsum* function (Cumulative Sums) was used, available in the standard libraries of the Statistical Data Analysis *R* [42], version 3.1.2 of 10/31/2014.

The remaining of the tasks described in the method for the calculation Rm and the mean of the R_m/S_m were obtained by conventional mathematical operations and functions.

Table I summarizes the values of *t*, *M*, R/S_t, $x = \log\{R/S_t\}$ and $y = \log\{t\}$, for the log-return of the historical prices of the asset FOREX AUD-JPY, in intervals of 1 day, where log is the e base logarithm, and (*x*, *y*) are the points of the graph plotted with scatter values of $\log\{R/S_t\}$, shown in Fig. 3.

Fig. 3 exhibits the scatter plot of the values of $\log\{R/S_t\}$ in relation to the values of $\log\{t\}$, for the relationship between AUD-JPY currencies, in intervals of 1 day. Fig. 3 also shows the trend line obtained through the *lm* (Fitting Linear Models) and *abline* (Add Lines to a Plot) functions, available in the standard libraries of the Statistical Data Analysis R [42].

The values for the estimation of the intercept $\hat{b}$ and the slope, for the line of the graph contained in the Fig. 3, are respectively -0.2481167 and 0.5534713. This slope represents the estimated value of the Hurst exponent (H). These values were obtained by the *lm* (Fitting Linear Models), *summary* (Summary of the Results of Model Fittings) and *coef* (Extracts Model Coefficients from Modeling Function) functions, also available in the standard libraries of the Statistical Data Analysis *R* [42].

The significance test, as suggested by de Couillard and Davison [7], using the statistic $p\text{–}value < 0.001$, is equal to 9.345983e-06. This value is obtained by the *t.test* (Student's t-Test) function, available in the standard libraries of the Statistical Data Analysis *R* [42].

C. Results

Table II summarizes the results obtained, where *H* is the estimated value for the Hurst Exponent, *I* is the *Intercept*, *t* is the *t-statistic* and *p* is the *p.value*.

As observed by Mandelbrot [4], who developed his studies based on Hurst's works [5], a Brownian motion has $H = ½$, while a persistent process has $H > ½$ and an anti-persistent process has $H < ½$. Therefore, from table II, is possible to confer that the historical series of prices for the pairs of currency prices, presented in this table, are shown as persistent processes, that is, $H > ½$, except for the parity between the currencies EUR-CHF, where it was not possible to confirm the Hurst Exponent in periods of 1 hour and 15 minutes. This is due to the fact that this series have data with low volatility between the data prices for periods of 1 hour and 15 minutes, and that the prices in these last two periods are almost constant.

Another fact, highlighted in the table II, containing the summarized results, is that the Hurst Exponent was constant between the intervals of 1 day, 1 hour and 15 minutes, which shows that the Hurst Exponent can be considered valid for the historical series presented in this table.

Fig. 4 displays the graphs of the historical series of prices for the pairs of currency prices: AUD-JPY, CHF-JPY, EURJPY and GBP-JPY, in periods of 1 day, 1 hour and 15 minutes, according to the periods presented in table 4.

Lam [14] suggests that time series can be examples of fractals. One of the questions proposed by this paper was to verify if the financial time series could use the properties of fractals, in particular the property of self-similarity, where parts of an object or process is similar to the object [43].

Fig. 4 shows that the similarity property, present in the fractals, is observed in the graphs of the historical series of prices for the pairs of currency prices: AUD-JPY, CHF-JPY, EUR-JPY and GBP-JPY, in periods of 1 day, 1 hour and 15 minutes, according to the periods presented in table II, which can corroborate with the proposition of Lam [14].
Fig. 5 demonstrates the values of log{R/S_t} in relation to the values of log{t} for the historical series of prices for the pairs of currency prices: AUD-JPY, CHF-JPY, EUR-JPY and GBP-JPY, in periods of 1 day, 1 hour and 15 minutes, according to the periods presented in table II.

TABLE II: VALUES OBTAINED ACCORDING TO THE METHODOLOGY DEVELOPED BY MANDELBROT AND WALLIS [4] BASED ON THE WORKS OF HURST [5].

Time Series	***Intervals***	***H***	***I***	***t***	***p***
AUD-JPY	1 day	0.57	-0.26	7.0	9.8e-6
	1 hour	0.56	-0.30	6.9	1.1e-5
	15 min.	0.54	-0.19	7.3	6.0e-6
AUD-JPY	1 day	0.57	-0.29	6.9	1.1e-5
	1 hour	0.56	-0.29	6.9	1.1e-5
	15 min.	0.55	-0.27	7.1	8.3e-6
EUR-CHF	1 day	0.55	-0.44	21.7	1.4e-11
	1 hour	0.17	1.40	53.2	1.3e-16
	15 min.	0.27	0.59	39.0	7.4e-15
EUR-JPY	1 day	0.57	-0.31	6.9	1.1e-5
	1 hour	0.54	-0.22	7.1	7.9e-6
	15 min.	0.57	-0.30	6.9	1.1e-5
GBP-JPY	1 day	0.58	-0.32	6.8	1.2e-5
	1 hour	0.56	-0.28	6.9	1.0e-5
	15 min.	0.53	-0.17	7.8	2.8e-6

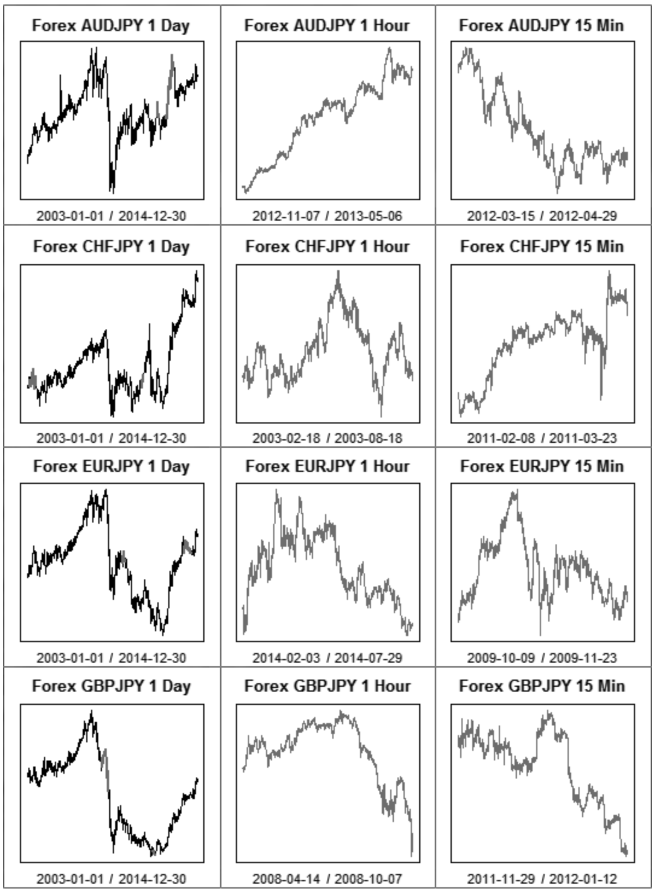

Fig. 4. Historical series for the pairs of currency AUD-JPY, CHF-JPY, EUR-JPY and GBP-JPY, with 1 day, 1 hour and 15 minutes intervals.

VII. CONCLUSION

This paper emphasized the applicability of Hurst Exponent (H), adopting the R/S Analysis, in the classification of the time series, particularly FOREX Securities, by giving a concrete information on correlation and persistence, certifying that this exponent is an excellent index for studying complex processes such as the financial time series.

Additionally, the paper reviewed the Fractal Theory in relation to the traditional Theory of Capital Markets, arguing about the hypothesis of the existence of a Fractal Market.

The inability to predict recurrent economic crises and the ineptitude to avoid the large losses resulting from this fact, jeopardize the efficiency of the current model of risk management, in this way evidencing the necessity to perfect the models. From this perspective, the fractal analysis emerges as an appropriate alternative, since it offers greater robustness to the elements of risk management, through more admissible assumptions.

There is already enough empirical evidence on the presence of the fractals, particularly in financial time series, and the more consistent perception of the Financial Market oscillations, recognizing patterns of persistence in the historical series, which allows the creation of effective policies in the identification of market risk.

Therefore, the initial assumption that the fractal behavior of the markets is capable of allowing an improved modeling of the Efficient Markets Hypothesis to explain the price movements has been ratified.

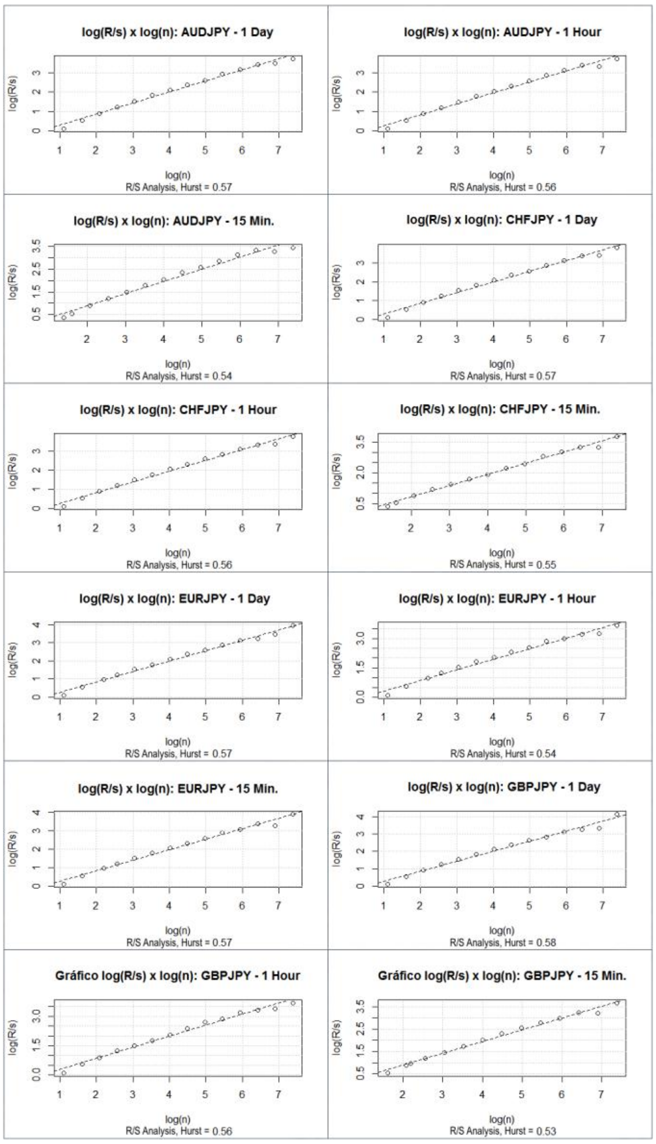

Fig. 5. Graphic with the values of log{R/S_t} in relation to the values of log{t} for the pairs of currency AUD-JPY, CHF-JPY, EUR-JPY and GBPJPY, with 1 day, 1 hour and 15 minutes intervals.

Added to this is a recommendation to market participants to address carefully with this theme, reviewing their bases and promoting the improvement of the methodology used to control the risks assumed.

However, it is noticed that the researches are still in the initial stage of the hypothesis corroboration, in which different conceptual approaches are used with the application of several statistical methods, with some results clearly indicating the presence of fractals in the financial time series.

Even with the confirmation of the assumption, the present study does not exhaust the subject, and opens space for discussion and development of new academic researches to ensure that it reaches maturity and can realize its full potential. Some research may increase the contributions of this work, such as comparative studies between hybrid models using Hurst Exponent (H) and Machine Learning or comparatives studies between traditional mathematical models, such as Autoregressive Integrated Moving Average (ARIMA) and Autoregressive Fractionally Integrated Moving Average (ARFIMA) models, or researches with others time series, such as commodities, appear as relevant themes of future research.

References

[1] F. G. Lima, F. A. C. Silva, and V. A. Antonini, “Análise do expoente de hurst em séries temporais financeiras,” *XII SEMEAD*, Vol. 1, 2009.

[2] P. Morettin, *Econometria Financeira: Um Curso em Séries Temporais Financeiras*, 2nd ed. Blucher, 2011.

[3] R. Racine, “Estimating the Hurst Exponent,” pp. 1–30, 2011.

[4] B. B. Mandelbrot and J. R. Wallis, “Robustness of the rescaled range R/S in the measurement of noncyclic long run Statistical dependence,” *Water Resources Research*, vol. 5, pp. 967, 1969.

[5] J.-F. Gouyet, “Physics and fractal structures,” *American Journal of Physics*, Vol. 65, no. 1, pp. 676, 1997.

[6] E. E. Peters, *Fractal Market Analysis: Applying Chaos Theory to Investment and Economics*, John Wiley and Sons, Inc., 1994.

[7] M. Couillard and M. Davison, “A comment on measuring the hurst exponent of financial time series,” *Physica A: Statistical Mechanics and its Applications*, vol. 348, no. 348, pp. 404–418, 2005.

[8] B. Qian and K. Rasheed, “Hurst exponent and financial market predictability,” 2004.

[9] M. Corazza and A. Malliaris, “Multifractality in foreign currency markets,” vol. 6, 2008.

[10] D. O. Cajueiro and B. M. Tabak, “Ranking efficiency for emerging markets,” *Chaos, Solitons and Fractals*, vol. 22, no. 2, pp. 349–352, 2004.

[11] C. Los and J. M. Lipka, “Long-term dependence characteristics of European stock indices,” 2004.

[12] J. P. Singh and S. Prabakaran, “On the distribution of returns and memory effects in Indian capital markets,” vol. 14, 2008.

[13] N. A. Kyaw, C. A. Los, and S. Zong, “Persistence characteristics of Latin American financial markets,” *Journal of Multinational Financial Management*, vol. 16, no. 3, pp. 269–290, 2006.

[14] L. Lam, *Introduction to Nonlinear Physics*. Springer, 2003.

[15] F. Family, “Book review: Fractal growth phenomena,” *Journal of Statistical Physics*, vol. 66, pp. 683–686, 1992.

[16] D. Luiz and H. Costa, “Geometria fractal,” *Revista Eletrônica de Educação e Ciência (REEC)*, vol. 4, no. 1, pp. 1–9, 2014.

[17] T. A. D. Assis *et al*., “Geometria fractal: Propriedades e Características de fractais ideais,” *Revista Brasileira de Ensino de Física*, vol. 30, pp. 2304.1–2304.10, 2008.

[18] K. Falconer, *Fractal Geometry: Mathematical Foundation and Applications*, John Wiley and Sons, Inc., 2003.

[19] B. B. Mandelbrot, *The Fractal Geometry of Nature*, W.H. Freeman and Company, 1983, vol. 51.

[20] E. E. Peters and D. A. Hsieh, *American Finance Association Chaos and Order in the Capital Markets: A New View of Cycles, Prices, and Market Volatility*. John Wiley and Sons, Inc., 2014, Vol. 48.

[21] M. F. M. Osborne, *Brownian Motion in the Stock Market*. P. H. Cootner - MIT Press, 1959, Vol. 7.

[22] E. F. Fama, “The behavior of stock-market prices,” *The Journal of Business*, vol. 38, pp. 34, 1965.

[23] W. Sharpe, *Portfolio Theory and Capital Markets*, McGraw-Hill, 1970.

[24] B. B. Mandelbrot, “The variation of certain speculative prices,” *The Journal of Business*, vol. 36, pp. 394, 1963.

[25] B. W. Ambrose, E. W. Ancel, and M. D. Griffiths, “Fractal structure in the capital markets revisited,” *Financial Analysts Journal*, vol. 49, pp. 73–77, 1993.

[26] U. A. Muller *et al*., “Fractals and Intrinsic Time: A Challenge to Econometricians,” *International Conference of the Applied Econometrics Association (AEA)* - XXXIX, 1993.

[27] M. Corazza, A. G. Malliaris, and C. Nardelli, “Searching for fractal structure in agricultural futures markets," *Journal of Futures Markets*, Vol. 17, pp. 433–473, 1997.

[28] M. Larrain, “Testing chaos and nonlinearities in T-Bill rates,” *Financial Analysts Journal*, vol. 47, pp. 51–62, 1991.

[29] M. K. P. So, “Long-term memory in stock market volatility,” *Applied Financial Economics*, vol. 10, no. 5, pp. 519–524, 2000.

[30] J. Barkoulas, W. C. Labys, and J. Onochie, “Fractional dynamics in international commodity prices,” *Journal of Futures Markets*, vol. 17, pp. 161–189, 1997.

[31] G. R. Richards, “The fractal structure of exchange rates: Measurement and forecasting,” *Journal of International Financial Markets, Institutions and Money*, Vol. 10, pp. 163–180, 2000.

[32] E. Panas, "Long memory and chaotic models of prices on the London metal exchange," *Resources Policy*, vol. 27, pp. 235–246, 2001.

[33] T. Di Matteo, T. Aste, and M. M. Dacorogna, "Long term memories of developed and emerging markets: Using the scaling analysis to characterize their stage of development," *J. Bank. Financ*, vol. 29, pp. 827–851, 2005.

[34] B. B. Mandelbrot, *Misbehavior of Markets*, 1st ed., B. Books, 2004.

[35] B. B. Mandelbrot, "Long-run linearity, locally Gaussian process, H-Spectra and infinite variances," *International Economic Review*, vol. 10, p. 82, 1969.

[36] B. G. Malkie, *A Random Walk Down Wall Street*, Editora Princeton University Press, 1990.

[37] V. Teverovsky, M. Taqqu, and W. Willinger, "A critical look at Lo's modified R/S statistic," *Journal of Statistical Planning and Inference*, vol. 80, no. 1, pp. 211–227, 1999.

[38] Free Forex Historical Data. *HistData.com*. (2017). [Online]. Available: http://www.histdata.com/

[39] D. Glassman, "The foreign exchange market," *Journal of International Economics*, Vol. 30, no. 3-4, pp. 385–387, 1991.

[40] Bis, "Triennial central bank survey. Foreign exchange turnover in April 2013: Preliminary global results," *Bank for International Settlements*, no. April, p. 24, 2013. [Online]. Available: www.bis.org

[41] A. Hejlsberg, "Visual studio .Net framework 4.6.2," 2016.

[42] R. Gentleman and R. Ihaka, "The R foundation for statistical computing," 2017.

[43] B. B. Mandelbrot, *Multifractals and 1/F Noise*. Springer, 1999, vol. 38.

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