Mathematical Model for Strap-on Boosters Launcher, Performances Evaluation

Teodor-Viorel Chelaru and Adrian Chelaru

***Abstract*—The paper presents some aspects regarding the mathematical model and performance evaluation for a two stages strap-on boosters launcher. This work uses two separate models dedicated for each flight phase. For the ascending phase, we will use a three degrees of freedom model in quasi–velocity frame. For the orbital phase, we will use a Gauss perturbing model. The results analysed will be in quasi-velocity frame but also some orbital parameters will be presented. Using these models, the strap-on boosters launcher performances will be evaluated. The novelty of the paper consists in orbital injection approach, with optimal manoeuvre description**

***Index Terms*—Mathematical model, orbital injection, strap-on booster launcher performances.**

I. Introduction

Today space programs are one of the priorities of the European Union. In order to ensuring the space access, Europe already has 3 launchers: Vega, Ariane and Soyuz, and intends to also develop the small and micro launcher class. To address the problem of designing a new launcher, the first step is to developed a mathematical model for performance evaluation, model which must be validated in a known case. In this idea, the paper proposes a performance evaluation model of a strap-on booster launcher, which is tested for Ariane 6 case. To approach this problem and in general for evaluating the launching capabilities it is necessary to elaborate an adequate mathematical model that ensures the evaluation of the launcher's capability to inject the payload on different circular orbits. The mathematical model presented below seeks to answer these needs. Having these requirements in mind, in order to develop the strap-on booster launcher (SBL) model, we will describe the necessary frames, the coordinate transformations, the equations of motion and the guidance law necessary to define the launcher motion for both flight phases.

II. Coordinate Systems

First, we will define the coordinate systems specific for the motion of the small launcher.

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A. The Earth Frame

This inertial coordinate system has the origin in the Earth's center and does not participate in its diurnal rotation (Earth spin). The axis X_p is in the equatorial plane along the vernal axis. The Z_p axis is along the polar axis, towards the North Pole. The Y_p axis is also in the equatorial plane and completes a right frame being in the equatorial plane.

B. The Local Frame

This coordinate system has the origin in the starting position, being earthbound, and participating in the diurnal rotation (Earth spin). The Y_L axis is the position along the **r** vector at the start moment. The Z_L axis is parallel with the equatorial plane, being oriented to the East. The X_L axis arising is forming with the first two axes a right frame (Fig. 1).

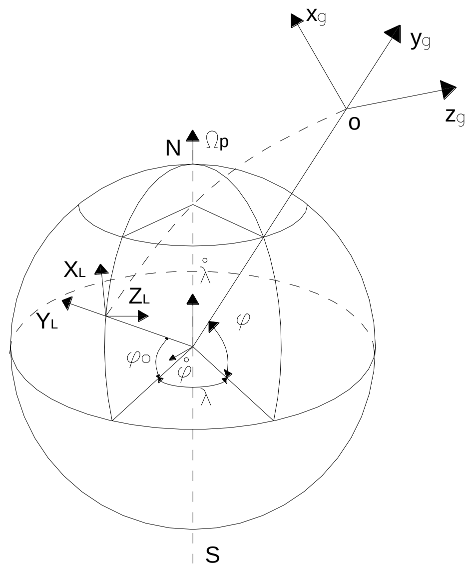

Fig. 1. The geocentric, local and geographical frames.

C. The Start Frame

This coordinate system has the origin in the starting position, being earthbound and participating in the diurnal rotation (Earth spin). The axis Y_S is the position along the **r** vector at the start moment. The axis X_S is oriented towards the launch direction and makes an azimuth angle ψ_0 with respect to the X_L axis. The Z_S axis, is forming with the first two axes a right frame, being oriented to the right with respect to the launch plane.

D. The Geographical Mobile Frame

This coordinate system has the origin in the mass center of

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the launcher, being earthbound and participating in the diurnal rotation. The axis y_g is the position along the $\mathbf{r}$ vector. The axis z_g is parallel with the equatorial plane, being oriented towards the East. The axis x_g is forming with first two axes a right trihedral. The geographical mobile frame overlaps the local frame at the start moment.

E. The Geocentric Spherical Frame

This coordinate system has the origin in the Earth's center, being earthbound and participating in its diurnal rotation (Earth spin). The launcher position can be described using spherical coordinates λ, ϕ, r (Fig. 1).

F. The Quasi-Velocity Frame

This coordinate system has the origin in the center of mass of the launcher. Similarly, to the velocity frame, the quasi-velocity frame has the axis x_a^* along the velocity vector, but the axis y_a^* it is in vertical plane. The axis z_a^* is forming with the first two axes a right trihedral. Next, we will use this trihedral to write the dynamic translation motion equations of the center of the mass.

III. The Gravitational Acceleration

If we consider the spherical Earth, the gravity is expressed by one term denoted g_{Ar} [2], oriented along radius r. This term, containing only the gravitational component without centrifugal contribution, which will be added later

$$g_{Ar} = \frac{\mu}{r^2}. \tag{1}$$

where:

$$\mu = 3.9861679 \cdot 10^{14} \tag{2}$$

IV. The Equations of Motion in Quasi — Velocity Frame

Because quasi-velocity frame is not an inertial frame, the dynamic equation of motion in quasi-velocity frame has following form [1], [3]:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{\Omega}_V^* \times \mathbf{V} = \frac{\mathbf{N}}{m} + \mathbf{g} + \mathbf{a}_c \tag{3}$$

where we have grouped the aerodynamic and thrust forces.

$$\mathbf{N} = \mathbf{F} + \mathbf{T} \tag{4}$$

The Coriolis acceleration is:

$$\mathbf{a}_c = -2\mathbf{\Omega}_p \times \mathbf{V} \tag{5}$$

The local derivative of the velocity in quasi-velocity frame is $\partial \mathbf{V}/\partial t$. $\mathbf{\Omega}_V^*$ is the rotation velocity of the quasi-velocity frame related to the local frame, which can be express as vectors:

$$\mathbf{\Omega}_V^* = \dot{\boldsymbol{\gamma}} + \dot{\boldsymbol{\chi}} + \dot{\boldsymbol{\phi}} + \dot{\boldsymbol{\lambda}} \tag{5}$$

The derivatives of latitude and longitude angles along geographical frame and the derivatives of the climb angle and the air-path track angle are presented in paper [1].

In this case, the components of the angular velocity vector along quasi-velocity frame become:

$$\begin{aligned} \omega_l^* = \dot{\lambda}\left(\cos\phi\cos\chi\cos\gamma + \sin\phi\sin\gamma\right) + \\ +\dot{\phi}\sin\chi\cos\gamma + \dot{\chi}\sin\gamma \end{aligned} \tag{6}$$

Taking in consideration that the vector $\mathbf{\Omega}_p$ has the same orientation as the vector $\dot{\boldsymbol{\lambda}}$, the Coriolis acceleration components in quasi-velocity frame are:

$$\begin{aligned} a_{cx} = 0;\ a_{cy} = -2V\Omega_{pz} = -2V\Omega_p \cos\phi \sin\chi; \\ a_{cz} = 2V\Omega_{py} = 2V\Omega_p\left(\sin\phi\cos\gamma - \cos\phi\cos\chi\sin\gamma\right) \end{aligned} \tag{7}$$

The gravitational acceleration previously introduced, is expressed by two terms, one term denoted g_r and oriented along radius r and the other term g_ω parallel with polar axis $N-S$. These two terms contain gravitational components and also centrifugal components given by the Earth's spin.

$$g_r = g_{Ar} - \Omega_p^2 r; \quad g_\omega = \Omega_p^2 r \sin\phi, \tag{8}$$

where g_{Ar} are given by relations (1) , (2), depending on the range.

Next, we will project the terms given by relation (8) along quasi-velocity frame. Summarizing, starting from relation (3), we obtain the dynamic equation which describes the motion of the center of mass of the launcher in quasi-velocity frame [1], [3]:

$$\begin{aligned} &\dot{V} = \frac{N_x}{m} - g_r \sin\gamma - g_\omega\left(\cos\phi\cos\chi\cos\gamma + \sin\phi\sin\gamma\right) \\ &\dot{\gamma} = \frac{N_y}{mV} - \frac{g_r}{V}\cos\gamma - \frac{g_\omega}{V}\left(-\cos\phi\cos\chi\sin\gamma + \sin\phi\cos\gamma\right) + \\ &+\frac{V}{r}\cos\gamma - 2\Omega_p\cos\phi\sin\chi \\ &\dot{\chi} = -\frac{N_z}{mV\cos\gamma} + \frac{g_\omega\cos\phi\sin\chi}{V\cos\gamma} + \frac{V}{r}\tan\phi\sin\chi\cos\gamma + \\ &+2\Omega_p(\cos\phi\cos\chi\tan\gamma - \sin\phi) \end{aligned} \tag{9}$$

Complemented with kinematic equations:

$$\dot{r} = V\sin\gamma .\ \dot{\phi} = \frac{V}{r}\cos\chi\cos\gamma;\ \dot{\lambda} = -\frac{V\sin\chi\cos\gamma}{r\cos\phi}; \tag{10}$$

where N_x, N_y, N_z are projection of the applied forces along quasi-velocity frame.

Supposing the aerodynamic angles are very small, the components of the applied forces become:

$$N_x = -D + X^T, N_y = Y^T, N_z = Z^T \tag{11}$$

where $X^T;Y^T;Z^T$ are the thrust components and D is the drag force.

Considering that the roll commands are given by separate Reaction Control System (RCS) and pitch δ_n and yaw δ_m commands are given through the angular deflection of the main Solid Rocket Motor (SRM), the thrust components are given by:

$$X^T = T\cos\delta_n\cos\delta_m; Y^T = -T\sin\delta_n\cos\delta_m; Z^T = T\sin\delta_m; \quad (12)$$

where:

$$\delta_n = k_1(\gamma - \gamma_d) + k_2 a_y; \quad \delta_m = 0; \quad (13)$$

With imposed value for climb angle γ_d, and

$$a_y = Y^T / m; \quad (14)$$

V. Evolution in Orbital Phase, Payload Injection

In order to evaluate the orbital phase and the payload injection we use as inertial frame, the Earth frame. First, we obtain the velocity in geographic frame with respect to the inertial frame, by adding Earth rotation:

$$\begin{aligned} V_{xg} &= V\cos\gamma\cos\chi; V_{yg} = V\sin\gamma; \\ V_{zg} &= -V\cos\gamma\sin\chi + r\Omega_p\cos\phi \end{aligned} \quad (15)$$

Hence:

$$v = \sqrt{V_{xg}^2 + V_{yg}^2 + V_{zg}^2} \;\; \gamma_i = \arcsin\left(V_{yg}/v\right) \quad (16)$$

Next, we are interested in the angle α between the range vector $\mathbf{r}$ and the absolute velocity vector $\mathbf{v}$ at the end of the ascending phase and beginning of the orbital phase [2]. Having the γ_i angle, we can write a simple relation:

$$\alpha = \pi/2 - \gamma_i \quad (17)$$

which allows the computing of the α angle. The other two values, v and r at the end of the ascending phase depend mainly on the launcher's characteristics: thrust and mass which are being obtained with equations (9), (10), (17) As it is shown in work [4], the knowledge of these three parameters at the end of the ascending phase is enough for the fully definition of the launcher's movement in the orbital phase. Using the Kepler model, one can determine the orbit elements. Thus, we can obtain immediately the kinetic moment and the unitary energy:

$$h = rv\sin\alpha \;\; E = v^2/2 - \mu/r \quad (18)$$

From where we get the parameter p, the geometrical elements of the orbit: e - eccentricity, a -semi-major axis end ψ -eccentric anomaly:

$$p = h^2/\mu \;\; e = 1 + 2Eh^2\mu^{-2} \;\; a = p/\left(1-e^2\right) \quad (19)$$

$$\cos\psi = \frac{a-r}{ae}$$

In order to obtain a circular orbit, from Gauss perturbing equations [2], we can extract the eccentricity equation:

$$\dot{e} = \frac{\zeta(\zeta^2 - f^2)a_T\cos\delta_2}{neaf}\left(\frac{e\zeta}{\zeta^2 - f^2}\sin\psi\sin\delta_1 + \cos\delta_1\right) \quad (20)$$

where: $\zeta^2 = 1 - e^2$; $f = 1 - e\cos\psi$

a_T - The acceleration derived from thrust.

δ_1 - The angular deflection of the thrust vector, relative to the perpendicular direction on $\mathbf{r}$ in the orbit plane;

δ_2 - The angular deflection of the thrust vector outside the orbit plane.

If we want an optimal maneuver to minimize in minimum time the eccentricity and achieve a circular orbit, we impose the following condition:

$$\frac{\partial\dot{e}}{\partial\delta_1} = \frac{\zeta(\zeta^2 - f^2)a_T\cos\delta_2}{neaf}\left(\frac{e\zeta}{\zeta^2 - f^2}\sin\psi\cos\delta_1 - \sin\delta_1\right) = 0 \quad (21)$$

And obtain an optimal value for the thrust angular deflection:

$$\tan\delta_1 = \frac{e\zeta}{\zeta^2 - f^2}\sin\psi \quad (22)$$

where e - eccentricity and ψ - eccentric anomaly.

Next, we evaluate the sign of relation (21) for the angular deflection (23) in order to obtain an eccentricity minimization.

If we substitute in relation (21) the angular deflection form (23) and consider the second angular deflection null $\left(\delta_2 = 0\right)$, we get:

$$\dot{e} = \frac{\zeta(\zeta^2 - f^2)a_T\cos\delta_1}{neaf}\left[\left(\frac{e\zeta}{\zeta^2 - f^2}\sin\psi\right)^2 + 1\right] \quad (23)$$

If we want to decrease the eccentricity, we impose the condition:

$$\zeta^2 - f^2 \le 0 \quad (24)$$

Equivalent with:

$$\cos^2\psi - 2e^{-1}\cos\psi + 1 > 0 \quad (25)$$

which means:

$$-1 \le \cos\psi \le (1-\zeta)e^{-1} \quad (26)$$

Based on these results (23) we can impose optimal pitch and yaw command for injection in circular orbit:

$$\delta_n = -\delta_1 + \gamma \; ; \; \delta_m = 0 \tag{27}$$

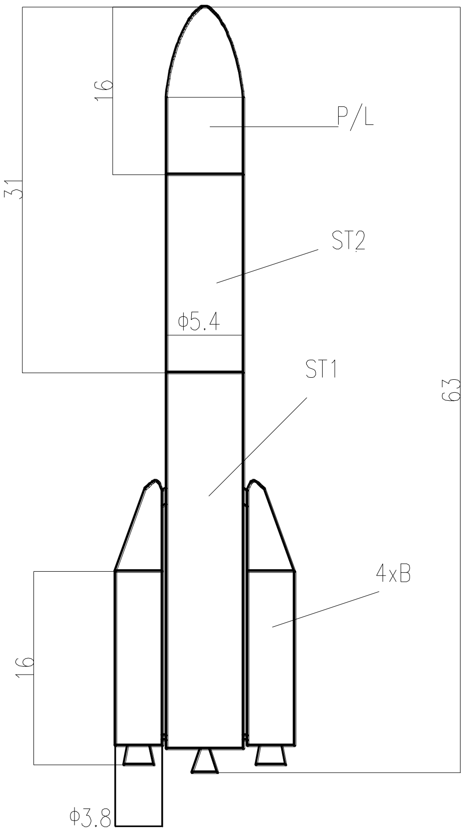

Fig. 2. SBL configuration.

VI. Optimizing the Ascending Phase

We start by describing for a two stage launcher with liquid fuel motor and four solid strap-on boosters, the typical ascending phase. Lift off is considerate from $t_0 = 0$ up to $t_1 = 2s$, when the climb angle is $\gamma = 90°$ and the SBL evolution is vertically. At $t_2 = 7s$ the climb angle is γ_1 and maintains this value up to $t_3 = t_3 + \Delta_1$. Between t_3 and t_4 ,the second ignition of second stage , the climb angle has no constrains, being in the gravity turn phase. Starting with second ignition of the second stage t_4 , the optimal manoeuvre is donned in order to make zero climb angle and also to cancel orbit eccentricity. The jettison fairing coincides with the separation of the stages 1 – 2. For SBL, the burning duration of the boosters is $t_{ab} = 94\,s$, the burnout duration of the first stage including booster burning duration is $t_{a1} = 534\,s$ and for the second stage is $t_{a2} = 760\,s$ (TABLE II) . We suppose that burnout duration of second stage separated in two phases with a duration $t_{a21} = \Delta_2$;$t_{a22} = 760\,s - \Delta_2$. Between the burnout of the first phase and the ignition of second phase we have a coasting phase with a duration Δ_3 . Summarizing, the ascending phase of SBL depends on four independent parameters, Δ_1, Δ_2, Δ_3 γ_1, which can be the subject of optimization. The strategy adopted consist that for different initial azimuth angle ψ_0 (orbit inclination) and different payload mass (MPL), to obtains by optimization: Δ_1, Δ_2, Δ_3, γ_1, which minimized performance index:

$$J = -\varepsilon_1 a + \int_0^{t_f} \left(\varepsilon_2 a_y^2 + \varepsilon_3 D\right) dt \tag{28}$$

where ε_k are the weights. We minimize them by using random number generators in a so called "Monte Carlo" method. The optimization method allows us to obtain at the end a circular orbit with maximum altitude and minimum manoeuvring effort for different orbit inclinations and different payload mass, which translates into SBL performances.

VII. Input Data for SBL Model

The input data used are taken from [5].

TABLE I: Mass Characteristics

Configuration	Mass [tons]	
	Initial	Final
4 Boosters + Stage I + Stage II + P/L+FER	809.6	299.7
Stage I + Stage II + P/L+FER	198.7	58.7
Stage II + P/L	40.9	11.
P/L	5	5

In Fig. 2 we have: P/L Payload;; ST - Stage; B - Boosters Main geometrical sizes at SBL start are: $l = 63m$ $d = 5.4m$

TABLE II: Thrust Characteristic

Stage	Specific impulse (*) [s]	Propellant mass [tons]	Burnout duration t [s]
B	280	480.	94
I	432	170.	534
II	465	30.	760

* Vacuum conditions

VIII. Test Case

As test case, we choose an equatorial orbit, with the following initial conditions: Geographic orientation: Azimuth angle $\psi_0 = 90°$ (towards the East); Geocentric latitude $\varphi = 0°$ (Equatorial latitude); Altitude: $h_0 = 1m$; Initial velocity $V_0 = 1[m/s]$; Initial climb angle $\gamma_0 = 90°$. Payload mass $MPL = 5000[kg]$. Corresponding to minimal value of performance index (29), we obtain: $\Delta_1 = 36[s]$,

$\Delta_2 = 378[s]$ $\Delta_2 = 260[s]$ $\gamma_1 = 86°$, which leads to a circular orbits with altitude $h_p = 2692[km]$. Using these parameters, we have defined a circular orbit described in next item.

IX. Results

Fig. 3 shows the relative velocity, which means the ratio between absolute velocity in inertial frame (17) and velocity corresponding to a circular orbit. We can observe that after injection phase relative velocity remain at unit value. In the same diagram is shown the altitude, which after injection remains at a constant value.

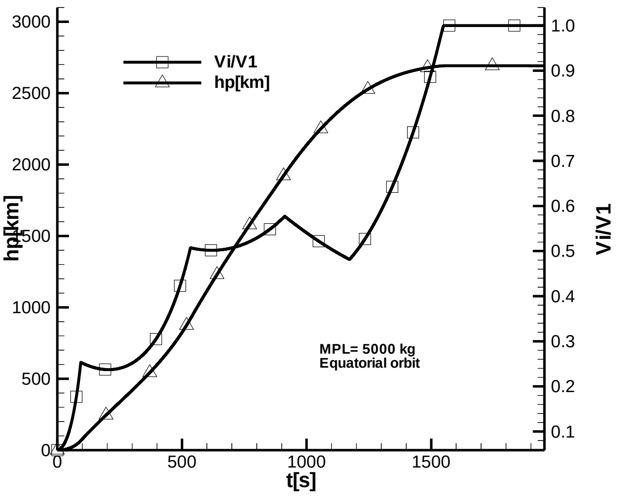

Fig. 3. Vi/V1 relative velocity and hp – Altitude.

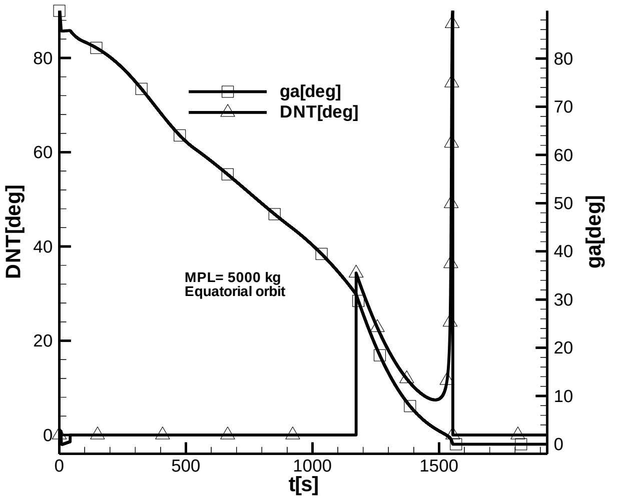

Fig. 4. DTN-Deflection angle and ga- climb angle.

Fig. 4 shows the deflection angle of the thrust vector related velocity during ascending and orbital phase. From this diagram, one can observe large values of the deflection angle during final injection manoeuvre. To solve this problem is necessary to rotate entire SBL body with a combined manoeuvre which uses in the same time TVC (Thrust Vector Control) and RCS (Reaction Control System). The same diagram shows the climb angle γ which is controlled by the thrust deflection angle. One can observe that it starts at $\gamma_0 = 90°$, followed by the imposed value $\gamma_1 = 86°$ and after orbital injection remains at zero value.

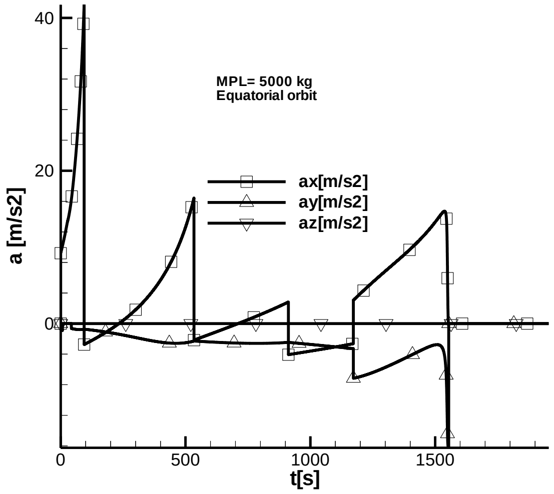

Fig. 5. Acceleration in quasi velocity frame

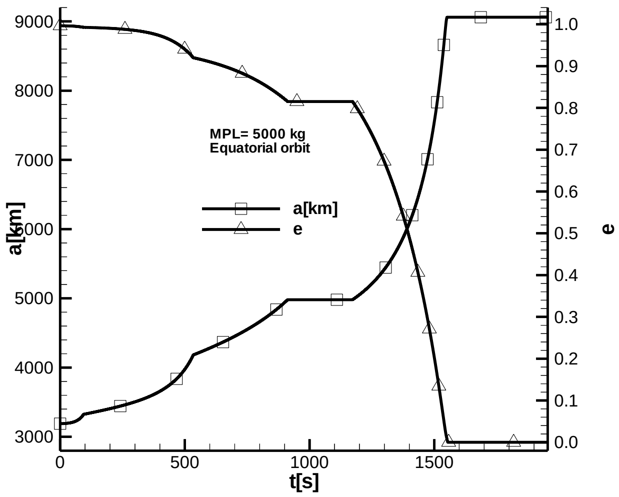

Fig. 6. E- eccentricity and a- semi-major axis.

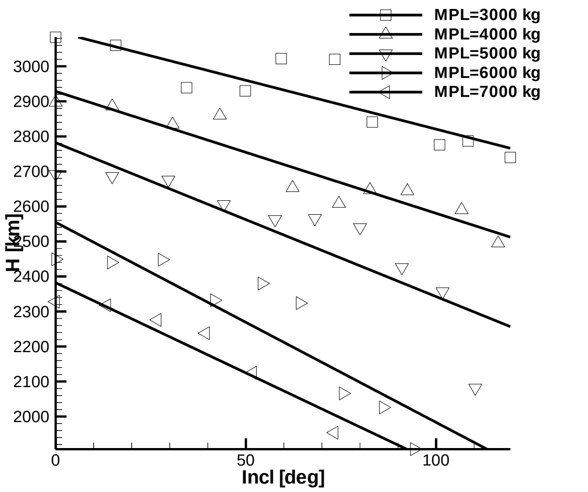

Fig. 7. Maximum altitude for circular orbit as function of inclination and payload mass.

For the same test case, Fig. 5 shows the acceleration in quasi - velocity frame. One can observe a_x as the result of thrust of each stage along velocity vector and also acceleration a_y normal on velocity in orbital plane. The acceleration a_z normal on orbital plane are insignificant.

Fig. 6 shows two orbital parameters, eccentricity and semi major axis during ascending and orbital phase.

One can observe that eccentricity decreases to zero, and after the injection phase remains at this value. In the same time the semi major axis increases simultaneous with velocity and remains constant after orbital injection.

Fig. 7 shows maximum altitude orbit as a function of orbit inclination and payload mass which means SBL performances.

X. Conclusions

As we said at the beginning, the paper has as objective the building of a simple mathematical model able to evaluate launcher's performances. In order to solve this problem, we separated the launcher's evolution in two phases, the first phase being the ascending phase until the launcher or the upper stage of it is in optimal position to make orbital injection and the second phase when the upper stage performs orbital manoeuvres and payload injection. For each phase, we developed a separate calculus model. For the ascending phase we developed a 3DOF model which describes the functionality of the launcher in the quasi-velocity frame in accordance with the work [2]. For the orbital phase, we used a sample model based on Kepler's theory [3], which allows us, to evaluate orbital parameters, and Gauss orbital perturbed equation [3] in order to obtain optimal injection manoeuvre. Despite different model used for each flight phases, for unitary approach we use actually only 3DOF model in quasi-velocity frame, by transform the command from orbital frame in quasi – velocity frame. Considering that launcher is targeting at circular orbits, we built a performance index based on maximum semi major-axis, minimum manoeuvring effort and minimum drag force, which allows the defining of the characteristics parameter of a trajectory required to obtain a circular orbit with maximum altitude at the end of orbital phase. The test case build and the results obtained prove the correctness of the developed model, including the strategy adopted for optimizing the accessional phase. Considering other case, with deferent initial condition, we used the model developed to evaluate the entire field of SBL performance. The solution adopted for SBL mission design must take into consideration that the accuracy of the desired orbit depends directly on the upper stage, which makes the injection for transferring the payload to the desired orbit.

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