Mixed Type Higher Order Symmetric Duality Over Cones

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Abstract—In this paper, a new mixed type higher-order symmetric duality in scalar programming over cone is formulated. The weak, strong and converse duality theorems are proved for these programs under \( \eta \)-invexity/\( \eta \)-pseudoinvexity assumptions. Self duality also discussed. As a special case of our duality relation, we give some known duality results. Our results generalize these existing dual formulations.

Index Terms—Higher-order symmetric duality, duality theorems, higher-order invexity/pseudoinvexity.

I. INTRODUCTION


Xu [8] formulated two mixed type duals in multiobjective programming and also proved duality theorems. Ahmad and Husain [9] studied mixed symmetric multiobjective dual programs and obtained duality results under K-preinvexity and K-pseudoinvexity assumptions. Chandra et al. [10] and Yang et al. [11] discussed a mixed symmetric dual formulation for a nonlinear programming problem and for a class of nondifferentiable nonlinear programming problems, respectively. Later on, Ahmad [12] formulated mixed type symmetric dual in multiobjective programming problems ignoring nonnegativity restrictions of Bector et al. [13].

In this paper, a new mixed type higher-order symmetric duality over cone in multiobjective programming is formulated. The weak, strong and converse duality theorems are proved for these programs under \( \eta \)-invexity/\( \eta \)-pseudoinvexity assumptions. Self duality also discussed. As a special case of our duality relation, we give some known duality results. Special cases are discussed to show that this study extends some of the known results in [14], [15] and [4].

II. PREREQUISITES

For \( N = \{ 1, 2, 3, \ldots, n \} \) and \( M = \{ 1, 2, 3, \ldots, m \} \), let \( J_1 \subset N, K_1 \subset M \) and \( J_2 = N \setminus J_1 \) and \( K_2 = M \setminus K_1 \). Let \( |J_1| \) denote the number of elements in the set \( J_1 \). The other numbers \( |J_2|, |K_1| \) and \( |K_2| \) are defined similarly. Notice that if \( J_1 = \emptyset \), then \( J_2 = N \), that is \( |J_1|=0 \) and \( |J_2|=n \). Hence, \( R^{|J_1|} \) is zero dimensional Euclidean space and \( R^{|J_2|} \) is n-dimensional Euclidean space. It is clear that any \( x \in R^n \) can be written as \( x = \{ x^1, x^2 \}, x^1 \in R^{|J_1|}, x^2 \in R^{|J_2|} \). Similarly, any \( y \in R^m \) can be written as \( y = \{ y^1, y^2 \}, y^1 \in R^{|K_1|}, y^2 \in R^{|K_2|} \).

We consider the following programming problem :

\[
\text{(P) Minimize } F(x), \quad -g(x) \in Q, \quad x \in S
\]

where \( S \subseteq R^{n+m} \) and \( F : S \to R \) and \( Q \) is a closed convex cone.

The following convention for vector inequalities will be used: If \( a, b \in R^n \), then

\[
a \preceq b \iff a_i \leq b_i, i = 1, 2, \ldots, n; \\
a \succeq b \iff a_i \geq b_i, i = 1, 2, \ldots, n; \\
a \succ b \iff a_i > b_i, i = 1, 2, \ldots, n.
\]

**Definition 2.1** A function \( \phi : S \mapsto R \) is said to be higher-order invex at \( u \in S \) with respect to \( \eta : S \times S \mapsto S \) and \( h : S \times S \mapsto R \), if for all \( (x, u) \in S \times S, \)

\[
\eta^T (x, u) [\nabla \phi(u)] + h(u, p) \geq \phi(x) - \phi(u) - h(u, p) + p^T \nabla h(u, p),
\]

for \( p \in S \times S \).

**Definition 2.2** A function \( \phi : S \mapsto R \) is said to be higher-order pseudoinvex at \( u \in R^n \) with respect to \( \eta : S \times S \mapsto S \) and \( h : S \times S \mapsto R \), if for all
\[(x, p) \in S \times S, \quad \eta^T(x,u)(\nabla_{\rho} h(u) + \nabla_{\rho} h(u, p)) \geq 0 \]

\[\Rightarrow \phi(x) - \phi(u) - h(u, p) + p^T \nabla_{\rho} h(u, p) \geq 0.\]

Unless otherwise stated, \(C_1, C_2, C_3\) and \(C_4\) represent closed convex cones in \(R_{[1]}^1, R_{[2]}^2, R_{[K]}^1\) and \(R_{[K]}^2\), respectively, with non-empty interiors and \(C_i^*, i = 1, 2, 3, 4\) is its polar cones and \(S_1 \subset R^n\) and \(S_2 \subset R^m\) are open sets such that \(C_1 \times C_2 \subset S_1 \times S_2\).

III. HIGHER-ORDER MIXED TYPE SYMMETRIC DUALITY

We consider the following pair of higher order symmetric duals and establish weak, strong and converse duality theorems.

Primal Problem (MHPC):

Minimize \(L(x, y, p) = f_1(x^1, y^1) + \int_2 f_2(x^2, y^2) + h_1(x^1, y^1, p^1) + h_2(x^2, y^2, p^2)\)

\[-(p^1)^T \nabla_{\rho} h_1(x^1, y^1, p^1) - (p^2)^T \nabla_{\rho} h_2(x^2, y^2, p^2)\]

\[-(y^1)^T [\nabla_{\rho} f_1(x^1, y^1) + \nabla_{\rho} h_1(x^1, y^1, p^1)]\]

Subject to

\[\nabla_{\rho} f_1(x^1, y^1) + \nabla_{\rho} h_1(x^1, y^1, p^1) \in C_3^*, \] (3.1)

\[\nabla_{\rho} f_2(x^2, y^2) + \nabla_{\rho} h_2(x^2, y^2, p^2) \in C_4^*, \] (3.2)

\[(y^1)^T [\nabla_{\rho} f_1(x^1, y^1) + \nabla_{\rho} h_1(x^1, y^1, p^1)] \geq 0, \] (3.3)

\[(p^1)^T [\nabla_{\rho} f_1(x^1, y^1) + \nabla_{\rho} h_1(x^1, y^1, p^1)] \geq 0, \] (3.4)

\[(p^2)^T [\nabla_{\rho} f_2(x^2, y^2) + \nabla_{\rho} h_2(x^2, y^2, p^2)] \geq 0, \] (3.5)

\[x^1 \in C_2, x^2 \in C_2, y^2 \geq 0, \] (3.6)

Dual Problem (MHDC):

Minimize \(L(x, y, p) = f_1(u^1, v^1) + f_2(u^2, v^2) + g_1(u^1, v^1, r^1) + g_2(u^2, v^2, r^2)\)

\[-(r^1)^T \nabla_{\rho} g_1(u^1, v^1, r^1) - (r^2)^T \nabla_{\rho} g_2(u^2, v^2, r^2)\]

\[-(u^1)^T [\nabla_{\rho} f_1(u^1, v^1) + \nabla_{\rho} g_1(u^1, v^1, r^1)]\]

Subject to

\[\nabla_{\rho} f_1(u^1, v^1) + \nabla_{\rho} g_1(u^1, v^1, r^1) \in C_3^*, \] (3.7)

\[\nabla_{\rho} f_2(u^2, v^2) + \nabla_{\rho} g_2(u^2, v^2, r^2) \in C_2^*, \] (3.8)

\[(u^1)^T [\nabla_{\rho} f_1(u^1, v^1) + \nabla_{\rho} g_1(u^1, v^1, r^1)] \leq 0, \] (3.9)

\[(r^1)^T [\nabla_{\rho} f_1(u^1, v^1) + \nabla_{\rho} g_1(u^1, v^1, r^1)] \leq 0, \] (3.10)

\[(r^2)^T [\nabla_{\rho} f_2(u^2, v^2) + \nabla_{\rho} g_2(u^2, v^2, r^2)] \leq 0, \] (3.11)

\[v^1 \in C_4, v^2 \in C_4, u^2 \geq 0, \] (3.12)

where

(i) \(f^1: R_{[1]}^1 \times R_{[K]}^1 \rightarrow R\),

(ii) \(f^2: R_{[2]}^2 \times R_{[K]}^2 \rightarrow R\),

(iii) \(g^1: R_{[1]}^1 \times R_{[K]}^1 \rightarrow R\),

(iv) \(g^2: R_{[2]}^2 \times R_{[K]}^2 \rightarrow R\),

(v) \(h^1: R_{[1]}^1 \times R_{[K]}^1 \times R_{[K]}^1 \rightarrow R\),

(vi) \(h^2: R_{[2]}^2 \times R_{[K]}^2 \times R_{[K]}^2 \rightarrow R\),

are twice differentiable functions, respectively,

(vii) \(p^1 \in R_{[K]}^1, p^2 \in R_{[K]}^2, r^1 \in R_{[1]}^1\) and \(r^2 \in R_{[2]}^2\).

IV. DUALITY THEOREMS

Theorem 4.1 (Weak Duality).

Let \((x^0, x^2, y^0, y^2, p^1, p^2)\) be feasible for (PP) and \((u^0, u^1, v^0, v^1, r^1, r^2)\) be feasible for (DP). Suppose that

(i) \(f^1(\cdot, \cdot, \cdot)\) is higher-order pseudo-invex at \(u^1\) with respect to \(\eta_1\) and \(g^1(u^1, v^1, r^1)\),

(ii) \(-f^1(x, \cdot, \cdot)\) is higher-order pseudo-invex at \(x^1\) with respect to \(\eta_2\) and \(-h^1(x^1, y^1, p^1)\),

(iii) \(f^2(\cdot, \cdot, \cdot)\) is higher-order invex at \(u^2\) with respect to \(\eta_3\) and \(g^2(u^2, v^2, r^2)\),

(iv) \(-f^2(x^2, \cdot, \cdot)\) is higher-order invex at \(y^2\) with respect to \(\eta_4\) and \(-h^2(x^2, y^2, p^2)\),

(v) \(\eta_1(x^1, u^1) + u^1 + r^1 \in C_1\),

(vi) \(\eta_2(v^0, y^0) + v^0 + p^1 \in C_1\),

(vii) \(\eta_3(u^2, u^2) + u^2 + r^2 \in C_2\),

(viii) \(\eta_4(v^2, y^2) + y^2 + p^2 \in C_4\).

Then

\[L(x^0, x^2, y^0, y^2, p^0, p^2) \geq M(u^0, u^1, v^1, v^2, r^1, r^2). \] (4.1)

Proof: From hypothesis (vii), (viii) and equations (3.2) and (3.8), we get

\[\eta_3(x^2, u^2)[\nabla_{\rho} f^2(u^2, v^2) + \nabla_{\rho} g^2(u^2, v^2, r^2)]\]

\[+u^2[\nabla_{\rho} f^2(u^2, v^2) + \nabla_{\rho} g^2(u^2, v^2, r^2)]\]

\[\geq-r^2[\nabla_{\rho} f^2(u^2, v^2) + \nabla_{\rho} g^2(u^2, v^2, r^2)],\]

\[\eta_4(v^2, y^2)[\nabla_{\rho} f^2(x^2, y^2) + \nabla_{\rho} h^2(x^2, y^2, p^2)]\]

\[+y^2[\nabla_{\rho} f^2(x^2, y^2) + \nabla_{\rho} h^2(x^2, y^2, p^2)]\]

\[\geq-p^2[\nabla_{\rho} f^2(x^2, y^2) + \nabla_{\rho} h^2(x^2, y^2, p^2)].\]
Which on using equations (3.5) and (3.11) implies that

\[
\eta_1(x^2, y^2)[\nabla_y^2 f^2(x^2, y^2) + \nabla_y^2 g^2(x^2, y^2, r^2)]
+ u^T[\nabla_y^2 f^2(u^2, v^2) + \nabla_y^2 g^2(u^2, v^2, r^2)] \geq 0. \tag{4.2}
\]

\[
\eta_1(v^2, y^2)[\nabla_y^2 f(x^2, y^2) + \nabla_y^2 h_2(x^2, y^2, p^2)]
+ y^2[\nabla_y^2 f(x^2, y^2) + \nabla_y^2 h_2(x^2, y^2, p^2)] \geq 0. \tag{4.3}
\]

Now from hypothesis (iii) and (iv), we have

\[
f^2(x^2, y^2) - f^2(u^2, v^2) - h^2(x^2, y^2, p^2)
+(p^2)^T \nabla_y^2 h^2(x^2, y^2, p^2) + h^2(u^2, v^2, p^2)
-(p^2)^T \nabla_y^2 h^2(x^2, y^2, p^2)
\geq \eta_1(x^2, u^2)[\nabla_y^2 f^2(u^2, v^2) + \nabla_y^2 g^2(u^2, v^2, r^2)],
\]

and

\[
\geq -u^T[\nabla_y^2 f^2(u^2, v^2) + \nabla_y^2 g^2(u^2, v^2, r^2)],
\]

\[
f^2(x^2, y^2) - f^2(x^2, y^2) - h^2(x^2, y^2, p^2)
+(p^2)^T \nabla_y^2 h^2(x^2, y^2, p^2) + h^2(x^2, y^2, p^2)
-(p^2)^T \nabla_y^2 h^2(x^2, y^2, p^2)
\geq \eta_1(v^2, y^2)[\nabla_y^2 f(x^2, y^2) + \nabla_y^2 h_2(x^2, y^2, p^2)],
\]

which along with equations (4.2) and (4.3), we obtain

\[
f^2(x^2, y^2) - f^2(u^2, v^2) - h^2(x^2, y^2, p^2)
+(p^2)^T \nabla_y^2 h^2(x^2, y^2, p^2) + h^2(u^2, v^2, p^2)
-(p^2)^T \nabla_y^2 h^2(x^2, y^2, p^2)
\geq -u^T[\nabla_y^2 f^2(u^2, v^2) + \nabla_y^2 g^2(u^2, v^2, r^2)],
\]

\[
f^2(x^2, y^2) - f^2(x^2, y^2) - h^2(x^2, y^2, p^2)
+(p^2)^T \nabla_y^2 h^2(x^2, y^2, p^2) + h^2(x^2, y^2, p^2)
-(p^2)^T \nabla_y^2 h^2(x^2, y^2, p^2)
\geq y^2[\nabla_y^2 f(x^2, y^2) + \nabla_y^2 h_2(x^2, y^2, p^2)].
\]

Now adding the above two equations, we get

\[
f^2(x^2, y^2) - h^2(x^2, y^2, p^2) + (p^2)^T \nabla_y^2 h^2(x^2, y^2, p^2)
-y^2[\nabla_y^2 f(x^2, y^2) + \nabla_y^2 h_2(x^2, y^2, p^2)]
\geq f^2(u^2, v^2) - h^2(u^2, v^2, p^2) + (p^2)^T \nabla_y^2 h^2(u^2, v^2, p^2)
- u^2[\nabla_y^2 f^2(u^2, v^2) + \nabla_y^2 g^2(u^2, v^2, r^2)]. \tag{4.4}
\]

Similarly, from hypothesis (i), (ii) and equations (3.1), (3.7), we get

\[
\eta_1(x^2, u^2)[\nabla_y^1 f^1(u^1, v^1) + \nabla_y^1 g^1(u^1, v^1, r^1)]
\geq -(u^1 + r^1)[\nabla_y^1 f^1(u^1, v^1) + \nabla_y^1 g^1(u^1, v^1, r^1)],
\]

\[
\eta_2(v^1, y^1)[\nabla_y^1 f^1(x^1, y^1) + \nabla_y^1 h_1(x^1, y^1, p^1)]
\geq -(y^1 + p^1)[\nabla_y^1 f^1(x^1, y^1) + \nabla_y^1 h_1(x^1, y^1, p^1)].
\]

Now inequalities (3.3), (3.4), (3.9) and (3.10) gives

\[
\eta_1(x^2, u^2)[\nabla_y^1 f^1(u^1, v^1) + \nabla_y^1 g^1(u^1, v^1, r^1)] \geq 0,
\]

and

\[
\eta_2(v^1, y^1)[\nabla_y^1 f^1(x^1, y^1) + \nabla_y^1 h_1(x^1, y^1, p^1)] \geq 0,
\]

which by hypothesis (i) and (ii) implies

\[
f^1(x^1, y^1) - f^1(u^1, v^1) - h^1(x^1, y^1, p^1)
+(p^1)^T \nabla_y^1 h^1(x^1, y^1, p^1)
+h^1(u^1, v^1, p^1) \geq 0, \tag{4.5}
\]

\[
f^1(x^1, y^1) - f^1(x^1, y^1) - h^1(x^1, y^1, p^1)
+(p^1)^T \nabla_y^1 h^1(x^1, y^1, p^1)
+h^1(u^1, v^1, p^1) \geq 0.
\]

Adding the above two inequalities, we get

\[
f^1(x^1, y^1) - h^1(x^1, y^1, p^1) + (p^1)^T \nabla_y^1 h^1(x^1, y^1, p^1)
\geq f^1(u^1, v^1) - h^1(u^1, v^1, p^1) + (p^1)^T \nabla_y^1 h^1(u^1, v^1, p^1).
\]

Combining inequalities (4.4) and (4.5), we have

\[
L(x^1, x^2, y^1, y^2, p^1, p^2) \geq M(u^1, u^2, v^1, v^2, r^1, r^2).
\]

Thus the results holds.

**Theorem 4.2 (Strong Duality).**

Let \((\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2, \bar{p}_1, \bar{p}_2)\) be an optimal solution of (MHPC). Suppose that

(i) \(\nabla_{\bar{p}_1}^1 h^1(\bar{x}_1', \bar{y}_1', \bar{p}_1')\) is positive or negative definite and \(\nabla_{\bar{p}_2}^2 h^2(\bar{x}_2', \bar{y}_2', \bar{p}_2')\) is negative definite,

(ii) \(\nabla_{\bar{x}_1} f^1(\bar{x}_1', \bar{y}_1') + \nabla_{\bar{y}_1} h^1(\bar{x}_1', \bar{y}_1', \bar{p}_1') \neq 0\)

and \(\nabla_{\bar{x}_2} f^2(\bar{x}_2', \bar{y}_2') + \nabla_{\bar{y}_2} h^2(\bar{x}_2', \bar{y}_2', \bar{p}_2') \neq 0\),

(iii) \((\bar{p}_1')^T \nabla_{\bar{x}_1} f^1(\bar{x}_1', \bar{y}_1') + \nabla_{\bar{y}_1} h^1(\bar{x}_1', \bar{y}_1', \bar{p}_1') = 0 \Rightarrow \bar{p}_1' = 0\)

and \(\bar{p}_2' = 0\).
\[ y^2[\nabla_{y^2} h^2(\tilde{x}^2, \tilde{y}^2, \tilde{p}^2)] - \nabla_{p^2} h^2(\tilde{x}^2, \tilde{y}^2, \tilde{p}^2) + \nabla_{y^2} f^2(\tilde{x}^2, \tilde{y}^2, \tilde{p}^2) = 0 \Rightarrow \tilde{p}^2 = 0, \]

\[
\begin{align*}
(4.11) & \quad \alpha \left[ (x^1, y^1, \tilde{x}^1, \tilde{y}^1, \tilde{p}^1) \right] = \beta^1 - \gamma \tilde{y}^1 - \delta^1 \tilde{p}^1 - \mu^1 (x^1 - \tilde{x}^1) \\
\geq 0, & \quad \text{forall } x^1 \in C_1, \\
(4.12) & \quad \alpha \left[ (x^2, y^2, \tilde{x}^2, \tilde{y}^2, \tilde{p}^2) \right] \beta^2 - \alpha^2 \tilde{y}^2 - \alpha^2 \tilde{p}^2 - \delta^2 \tilde{p}^2 \\
+ \alpha \left[ (x^2, y^2, \tilde{x}^2, \tilde{y}^2, \tilde{p}^2) \right] \beta^2 - \alpha^2 \tilde{y}^2 - \alpha^2 \tilde{p}^2 - \delta^2 \tilde{p}^2 \\
- \mu^2 (x^2 - \tilde{x}^2) \geq 0, & \quad \text{forall } x^2 \in C_2, \\
(4.13) & \quad \alpha \left[ (x^1, y^1, \tilde{x}^1, \tilde{y}^1, \tilde{p}^1) \right] = \beta^1 - \alpha \tilde{p}^1 - \gamma \tilde{y}^1 - \delta \tilde{p}^1 = 0, \\
\text{And} & \quad (\beta^2 - \alpha \tilde{p}^2 - \alpha \tilde{p}^2 - \delta \tilde{p}^2) = 0, \\
(4.14) & \quad \gamma(\tilde{x}^1, \tilde{y}^1, \tilde{p}^1) = 0, \\
(4.15) & \quad \delta^2 \tilde{p}^2[\nabla_{p^2} f^2(\tilde{x}^2, \tilde{y}^2) + \nabla_{p^2} h^2(\tilde{x}^2, \tilde{y}^2, \tilde{p}^2)] = 0, \\
(4.16) & \quad \mu^1 \tilde{x}^1 = 0, \\
(4.17) & \quad \mu^2 \tilde{x}^2 = 0, \\
(4.18) & \quad \tilde{z}^2 = 0, \\
(4.19) & \quad (\alpha, \beta^1, \beta^2, \gamma, \delta^1, \delta^2, \mu^1, \mu^2, \tilde{z}^2) \neq 0, \\
(4.20) & \quad (\alpha, \beta^1, \beta^2, \gamma, \delta^1, \delta^2, \mu^1, \mu^2, \tilde{z}^2) \geq 0, \\
(4.21) & \quad \text{Premultiplying equations (4.10), (4.11) by } (\beta^1 - \alpha \tilde{p}^1 - \gamma \tilde{y}^1 - \delta \tilde{p}^1), (\beta^2 - \alpha \tilde{p}^2 - \alpha \tilde{p}^2 - \delta \tilde{p}^2), \text{ respectively and then using equations (4.12)-(4.16), we get} \\
(4.22) & \quad \beta^1 = \alpha \tilde{p}^1 + \gamma \tilde{y}^1 + \delta \tilde{p}^1. \\
\end{align*}
\]
which on using hypothesis (i)

\[ \beta^2 = \alpha \phi^2 + \alpha \gamma^2 + \delta^2 \rho^2. \]  

(4.23)

From equations (4.10) and (4.11), and hypothesis (ii), we obtain

\[ \delta^1 = 0, \]  

(4.24)

And

\[ \delta^2 = 0. \]  

(4.25)

Now suppose, \( \alpha = 0 \). Then equations (4.23), with (4.9) gives \( \xi^2 = 0 \) and, with (4.25) implies \( \beta^2 = 0 \) also equations (4.6), (4.22) implies \( \mu^1 = 0 \) and equations (4.7), (4.23) and (4.25) implies \( \mu^2 = 0 \). From equation (4.8) and hypothesis (ii) yield \( \gamma = 0 \), which along with equation (4.22), (4.24) reduces \( \beta^1 = 0 \). Thus \( (\alpha, \beta^1, \beta^2, \gamma, \delta^1, \delta^2, \mu^1, \mu^2) = 0 \), a contradiction to (4.20).

Hence \( \alpha = 0 \).  

(4.26)

Using equations (4.12), (4.14) and (4.15), we have

\[
(\beta^1 - \gamma \xi^1 - \delta^1 \rho^1 \eta \xi^1 \rho^1 \eta)[\nabla_{\gamma^1} f^1(\xi^1, \xi^1)] \\
+ \nabla_{\rho^1} h(\xi^1, \xi^1, \rho^1)] = 0,
\]

(4.27)

which along with hypothesis (iii) yield

\[ \rho^1 = 0. \]  

(4.28)

Further, from equation (4.9) and (4.23), we get

\[
\alpha^2[\nabla_{\gamma^2} h^2(\xi^2, \xi^2, \rho^2)] - \nabla_{\rho^2} h(\xi^2, \xi^2, \rho^2) \\
+ \nabla_{\gamma^2} f^2(\xi^2, \rho^2)] - \delta^2 = 0.
\]

(4.29)

Now from hypothesis (iii), we obtain

\[ \rho^2 = 0. \]  

(4.30)

Therefore equation (4.22) and (4.23) reduce to

\[ \beta^1 = \bar{\beta}^1, \]  

(4.31)

And

\[ \beta^2 = \alpha \phi^2. \]  

(4.32)

Also, it follows from equations (4.8), (4.22), (4.28) and hypothesis (ii) and (iv) that

\[ \alpha = \gamma > 0. \]  

(4.33)

So equation (4.31) implies

\[ \bar{\gamma}^1 = \frac{\beta^1}{\gamma}. \]  

(4.34)

Moreover, equation (4.6), (4.7) along with (4.22), (4.23), (4.30) and hypothesis (iv) yields

\[
\alpha[\nabla_{\gamma^1} f^1(\xi^1, \xi^1)] + \nabla_{\rho^1} h(\xi^1, \xi^1, \rho^1) \\
- \mu^1](x^1 - \bar{x}^1) \geq 0, \text{ for all } x^1 \in C_1,
\]

(4.35)

\[
\alpha[\nabla_{\gamma^2} f^2(\xi^2, \xi^2)] + \nabla_{\rho^2} h(\xi^2, \xi^2, \rho^2) \\
- \mu^2](x^2 - \bar{x}^2) \geq 0, \text{ for all } x^2 \in C_2,
\]

(4.36)

Let \( x^1 \in C_1 \), then \( \bar{x}^1 + x^1 \in C_1 \) and then above inequality implies

\[
\alpha x''[\nabla_{\gamma^1} f^1(\xi^1, \xi^1)] + \nabla_{\rho^1} h(\xi^1, \xi^1, \rho^1)] \\
- \mu^1 \bar{x}^1 = 0, \text{ for all } x^1 \in C_1.
\]

Further by using equation (4.17) the above inequality also be rewritten as

\[
\bar{x}''[\nabla_{\gamma^1} f^1(\xi^1, \xi^1)] + \nabla_{\rho^1} h(\xi^1, \xi^1, \rho^1)] \\
\geq \mu^1 \bar{x}^1 = 0, \text{ for all } x^1 \in C_1.
\]

Therefore

\[ \nabla_{\gamma^1} f^1(\xi^1, \xi^1)] + \nabla_{\rho^1} h(\xi^1, \xi^1, \rho^1) \in C^*_1, \]  

(4.37)

Similarly, we also obtain that

\[ \nabla_{\gamma^2} f^2(\xi^2, \xi^2)] + \nabla_{\rho^2} h(\xi^2, \xi^2, \rho^2) \in C^*_2. \]  

(4.38)

Thus \( (\bar{x}^1, \bar{x}^2, \bar{\gamma}^1, \bar{\gamma}^2, \bar{\rho}^1 = 0, \bar{\rho}^2 = 0) \) satisfies the dual constraints (3.7)-(3.12), i.e, it is an feasible solution of (MHDC) .

Also, using hypothesis (iv) we get the values of the objective functions of (MHPC) and (MHDC) at

\[ (\bar{x}^1, \bar{x}^2, \bar{\gamma}^1, \bar{\gamma}^2, \bar{\rho}^1 = 0, \bar{\rho}^2 = 0) \]  

and

\[ (\bar{x}^1, \bar{x}^2, \bar{\gamma}^1, \bar{\gamma}^2, \bar{\rho}^1 = 0, \bar{\rho}^2 = 0) \]  

are equal. Using Weak duality it easily shown that
(\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{r}^1 = 0, \bar{r}^2 = 0) \quad \text{and} \\
(\tilde{x}^1, \tilde{x}^2, \tilde{y}^1, \tilde{y}^2, \tilde{p}^1 = 0, \tilde{p}^2 = 0) \text{are optimal solutions for (MHPC) and (MHDC), respectively.}

**Theorem 4.3** (Converse Duality).

Let \((\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{r}^1, \bar{r}^2)\) be an optimal solution of (MHDC). Suppose that

(i) \(\nabla_{\bar{r}_1, \bar{r}_2} g^1(\bar{u}^1, \bar{v}^1, \bar{r}^1)\) is positive or negative definite

and \(\nabla_{\bar{r}_2, 2} g^2(\bar{u}^1, \bar{v}^1, \bar{r}^1)\) is negative definite,

(ii) \(\nabla_{\bar{u}_1} f^1(\bar{u}^1, \bar{v}^1) + \nabla_{\bar{v}_1} g^1(\bar{u}^1, \bar{v}^1, \bar{r}^1) \neq 0\) and

\(\nabla_{\bar{r}^2} f^2(\bar{x}^2, \bar{y}^2) + \nabla_{\bar{p}^2} h^2(\bar{x}^2, \bar{y}^2, \bar{p}^2) \neq 0\)

(iii) \((\bar{r}^1)^T [\nabla_{\bar{u}_1} f^1(\bar{u}^1, \bar{v}^1) + \nabla_{\bar{v}_1} g^1(\bar{u}^1, \bar{v}^1, \bar{r}^1)] = 0 \Rightarrow \bar{r}^1 = 0\) and

\(\nabla_{\bar{u}_2} g^2(\bar{u}^2, \bar{v}^2) - \nabla_{\bar{r}^2} g^2(\bar{u}^2, \bar{v}^2)\)

\(+ \nabla_{\bar{r}_2} r^2 = 0 \Rightarrow \bar{r}^2 = 0\)

(iv) \(g^1(\bar{u}^1, \bar{v}^1, 0) = g^1(\bar{u}^1, \bar{v}^1, 0), \nabla_{\bar{u}_1} g^1(\bar{u}^1, \bar{v}^1, 0) = g^1(\bar{u}^1, \bar{v}^1, 0)\),

\(\nabla_{\bar{u}_2} g^2(\bar{u}^2, \bar{v}^2, 0) = h^2(\bar{u}^2, \bar{v}^2, 0), \nabla_{\bar{r}_2} r^2 = 0\).

Then

(1) \((\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{r}^1 = 0, \bar{r}^2 = 0)\) is feasible for (MHPC) and

(2) \(L(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{p}^1, \bar{p}^2) = M(\bar{u}^1, \bar{u}^2, \bar{v}^1, \bar{v}^2, \bar{r}^1, \bar{r}^2)\).

Furthermore, if the hypothesis of Theorem 4.1 are satisfied for all feasible solutions of (MHPC) and (MHDC), then

\((\bar{x}^1, \bar{x}^2, \bar{y}^1, \bar{y}^2, \bar{r}^1 = 0, \bar{r}^2 = 0)\) is an optimal solution for (MHPC).

**Proof.** Follows on the line of Theorem 4.2.

**Theorem 4.4** (Self Duality).

A primal (dual) problem having equivalent dual (primal) formulation is said to be self-dual, that is, if the dual can be recast in the form of the primal. In general, (MHPC) and (MHDC) are not self-duals without some added restrictions on \(f; g; h\). If we assume

\(f^1: R^{[\bar{u}]} \times R^{[\bar{r}]} \to R, \quad f^2: R^{[\bar{u}]} \times R^{[\bar{r}]} \to R, \quad g^1: R^{[\bar{u}]} \times R^{[\bar{r}]} \to R, \quad g^2: R^{[\bar{u}]} \times R^{[\bar{r}]} \to R, \quad h^1: R^{[\bar{u}]} \times R^{[\bar{r}]} \times R^{[\bar{r}]} \to R, \quad h^2: R^{[\bar{u}]} \times R^{[\bar{r}]} \times R^{[\bar{r}]} \to R\), to be skew symmetric, that is

\(f^i(u^i, v^i) = -f^i(u^i, v^i), i = 1, 2\), and

\(g^i(u^i, v^i, r^i) = -g^i(u^i, v^i, r^i), i = 1, 2\),

Then we shall show that (MHPC) and (MHDC) are self-duals. By recasting the dual problem (MHDC) as a minimization problem, we have

Minimize \(M(u, v, r) = \)

\(-\{f_1(u^i, v^i) + f_2(u^i, v^i) + g_1(u^i, v^i, r^i) + g_2(u^i, v^i, r^i)\}

\(-(r^i)^T \nabla_{\bar{u}_1} g_1(u^i, v^i, r^i) - (r^i)^T \nabla_{\bar{v}_1} g_2(u^i, v^i, r^i)\)

\(- (u^i)^T \nabla_{\bar{u}_2} f^2(u^i, v^i) + \nabla_{\bar{r}_2} g_2(u^i, v^i, r^i))\}

Subject to

\(\nabla_{\bar{u}_1} f_1(u^i, v^i) + \nabla_{\bar{v}_1} g_1(u^i, v^i, r^i) \geq 0, \nabla_{\bar{u}_2} f_2(u^i, v^i) + \nabla_{\bar{r}_2} g_2(u^i, v^i, r^i) \geq 0,\)

\((u^i)^T \nabla_{\bar{u}_1} f_1(u^i, v^i) + \nabla_{\bar{v}_1} g_1(u^i, v^i, r^i) \leq 0,\)

\((r^i)^T \nabla_{\bar{u}_2} f^2(u^i, v^i) + \nabla_{\bar{r}_2} g_2(u^i, v^i, r^i) \leq 0,\)

\(u^i, v^i, u^i, v^i \geq 0,\)

As \(f, g, h\) are skew symmetric, i.e.,

\(\nabla_{\bar{u}_1} f_1(u^i, v^i) = -\nabla_{\bar{u}_1} f_1(v^i, u^i), \nabla_{\bar{u}_2} f_2(u^i, v^i) = -\nabla_{\bar{u}_2} f_2(v^i, u^i), \nabla_{\bar{u}_1} g_1(u^i, v^i, r^i) = -\nabla_{\bar{u}_1} g_1(v^i, u^i, r^i), \quad \text{and} \quad \nabla_{\bar{u}_2} g_2(u^i, v^i, r^i) = -\nabla_{\bar{u}_2} g_2(v^i, u^i, r^i),\)

Then the above problem becomes:

Minimize \(M(u, v, r) = \)

\(f_1(v^i, u^i) + f_2(v^i, u^i) + g_1(v^i, u^i, r^i) + g_2(v^i, u^i, r^i)\)

\(+(r^i)^T \nabla_{\bar{u}_1} g_1(v^i, u^i, r^i) - (r^i)^T \nabla_{\bar{v}_1} g_2(v^i, u^i, r^i)\)

\(-(u^i)^T \nabla_{\bar{u}_2} f^2(v^i, u^i) + \nabla_{\bar{r}_2} g_2(v^i, u^i, r^i))\}

Subject to

\(\nabla_{\bar{u}_1} f_1(v^i, u^i) + \nabla_{\bar{v}_1} g_1(v^i, u^i, r^i) \leq 0, \nabla_{\bar{u}_2} f_2(v^i, u^i) + \nabla_{\bar{r}_2} g_2(v^i, u^i, r^i) \leq 0,\)

\((u^i)^T \nabla_{\bar{u}_1} f_1(v^i, u^i) + \nabla_{\bar{v}_1} g_1(v^i, u^i, r^i) \geq 0,\)

\((r^i)^T \nabla_{\bar{u}_2} f^2(v^i, u^i) + \nabla_{\bar{r}_2} g_2(v^i, u^i, r^i) \geq 0,\)

\(v^i, u^i, v^i, u^i \leq 0,\)
Which shows that $M(u,v,p)$ is identical to $L(x,y,p)$, that is, the objective and the constraint functions are identical. Thus, the problem $L(x,y,p)$ becomes self-dual.

It is obvious that the feasibility of $(x^1, x^2, y^1, y^2, p^1, p^2)$ for $L(x,y,p)$ implies the feasibility of $(y^1, y^2, x^1, x^2, p^1, p^2)$ for $(MHPC)$ implies the feasibility of for $(MHDC)$ and vice versa.

V. SPECIAL CASE

In this section, we consider some special cases of our problems by choosing particular forms of the closed convex sets $C_1$ and $C_2$. In all these cases, $h(x,y,p) = (1/2)p^T\nabla_y f(x,y)p$
and $g(u,v,r) = (1/2)r^T\nabla_v f(u,v)r$,

(a) If $|K_1| = 0$, $|J_1| = 0$, $p = 0$ and $r = 0$, then (MHPC) and (MHDC) reduce to the programs studied in Chandra and Kumar [14].

(b) If $|K_1| = 0$, $|J_1| = 0$, $C_1 = R^m_+$ and $C_2 = R^n_+$, then after removing inequalities (3.5), (3.11), our programs reduce to the problems considered in Mishra [4].

(c) If $|K_1| = 0$, $|J_1| = 0$, $C_1 = R^m_+$ and $C_2 = R^n_+$, then the then after removing inequalities (3.5), (3.11), programs reduce to the second-order symmetric dual programs of Gulati et al. [15].

VI. CONCLUSION

A pair of mixed symmetric dual programs has been formulated by considering the optimization under the assumptions of $\eta$-invexity and $\eta$-pseudoinvexity. It may be noted that the symmetric duality between (MHPC) and (MHDC) can be utilized to establish non-differentiable mixed symmetric duality in integer over cone and other related programming problems.

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