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On the Control of End Point Tensions in a Vibration Problem

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***Abstract*—This study aims to investigate the optimality conditions in an optimal control problem governed by a hyperbolic problem. In the problem of vibration of one dimensional wire, the minimization of the distance between the end points of wire and the desired target functions is carried out by controlling the tensions of end points.**

***Index Terms*—Optimal control, optimal solutions, second-order hyperbolic equations.**

I. Introduction

This paper deals with the problem

$$J_{\alpha^*} = \inf_{h\in H} J_\alpha(h) = J_\alpha(h^*) \tag{1}$$

By the functional

$$J_\alpha(h) = \beta_1\int_0^T \left[u(0,t)-y_1(t)\right]^2 dt + \beta_2\int_0^T\left[u(l,t)-y_2(t)\right]^2 dt + \alpha\|h\|_H^2 \tag{2}$$

On the admissible pairs set

$$h(t)\in H := \left\{h=\{f(t);g(t)\}:\ \|h(t)\|_H^2 \le \tilde{h}\right\} \tag{3}$$

Subject to the following one dimensional vibration problem on the domain $\Omega := (0,l)\times(0,T)$;

$$p(x)u_{tt} = (k(x)u_x)_x + F(x,t),\quad (x,t)\in\Omega \tag{4}$$

$$u(x,0)=\varphi_1(x),\ u_t(x,0)=\varphi_2(x),\ x\in(0,l) \tag{5}$$

$$k(0)u_x(0,t)=f(t),\ k(l)u_x(l,t)=g(t),\ t\in(0,T) \tag{6}$$

Here, the coefficient functions $p(x)$ and $k(x)$ (Young modulus) represent the density and rigidity of an one dimensional wire respectively and satisfy the following conditions

$$0<p_0\le p(x)\le p_1,\ 0<k_0\le k(x)\le k_1,\ \overset{\circ}{\forall} x\in(0,l) \tag{7}$$

Also, the initial data represent the initial status and initial velocity of the wire and taken from the spaces

$$\varphi_1(x)\in H^1(0,l),\ \varphi_2(x)\in L_2(0,l). \tag{8}$$

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The functions $f(t)$ and $g(t)$ mean the tensions (Hooke Law) at the end points and play the role of control functions.

The inner product and norm on the set H , which is sub space of $L_2(0,T)\times L_2(0,T)$, can be defined as

$$\langle h_1,h_2\rangle_H = \int_0^T\left[f_1(t)f_2(t)+g_1(t)g_2(t)\right]dt$$

and

$$\|h(t)\|_H^2 = \int_0^T\left[f^2(t)+g^2(t)\right]dt$$

For $\forall h_1 := \{f_1,g_1\}\in H,\ \forall h_2 := \{f_2,g_2\}\in H$ and $\forall h := \{f,g\}\in H$.

We are interested in the weak solution $u\in H^1(\Omega)$ with $u(x,0)=\varphi_1(x)$ and $u_t(x,0)=\varphi_2(x), x\in(0,l)$ of the problem (4)-(6) which satisfies the following integral equation;

$$\int_\Omega\left[-p(x)u_t\eta_t + k(x)u_x\eta_x\right]dxdt - \int_0^T g(t)\eta(l,t)dt + \int_0^T f(t)\eta(0,t)dt - \int_0^T\int_0^l F(x,t)\eta(x,t)dxdt = 0 \tag{9}$$

For $\forall\ \eta\in H^1(\Omega),\ \ \eta(x,T)=0.$

It can be seen in [1] that solution in the sense of (9) is exist, unique and satisfies the inequalities

$$\max_{0\le t\le T}\left(\|u(\cdot,t)\|^2_{L_2(0,l)} + \|u_t(\cdot,t)\|^2_{L_2(0,l)} + \|u_x(\cdot,t)\|^2_{L_2(0,l)}\right) \le c_0\left(\|\varphi_1\|^2_{H^1(0,l)} + \|\varphi_2\|^2_{L_2(0,l)} + \|F\|^2_{L_2(\Omega)} + \|h\|^2_H\right) \tag{10}$$

and

$$\|u\|^2_{H^1(\Omega)} \le c_1\left(\|\varphi_1\|^2_{H^1(0,l)} + \|\varphi_2\|^2_{L_2(0,l)} + \|F\|^2_{L_2(\Omega)} + \|h\|^2_H\right) \tag{11}$$

where c_0 and c_1 are independent from $\varphi_1,\ \varphi_2,\ F$ and h .

The functional $J_\alpha(h)$ is called cost functional with the given numbers β_1 and β_2 such as $\beta_1\ge 0,\ \beta_2\ge 0,\ \beta_1+\beta_2\ne 0$. The functions $y_1(t)\in L_2(0,T)$ and $y_2(t)\in L_2(0,T)$ are given functions. The number $\alpha>0$

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is regularization parameter and the third term in the cost functional is called regularization term; its role is to avoid using "too large" controls in the minimization of $J_\alpha(h)$.

So, we want to control the tensions of end points in an one-dimensional vibration problem, given by the functions $f(t)$ and $g(t)$, in order to minimize the distance between the end points $u(0,t)$, $u(l,t)$ and desired target functions $y_1(t)$, $y_2(t)$.

In [2]-[17], similar problems with different controls and cost functions have been investigated by some authors.

This study is contributive in view of the property of controllability of left and right end point tensions simultaneously.

The main issues we deal with in the considered optimal control problem are:

1) Establishing existence and uniqueness of an optimal pair $\{u^*, h^*\}$.
2) Deriving necessary and sufficient optimality conditions.
3) Constructing an algorithm for the numerical approximation of $\{u^*, h^*\}$.

II. Existence and Uniqueness of An Optimal Pair

The strategy to prove existence and uniqueness of an optimal control is to use the relationship between minimization of quadratic functional and variational problems corresponding to symmetric bilinear forms.

Principally, the key point is to rewrite the functional $J_\alpha(h)$ in the following way:

$$J_\alpha(h) = b(h,h) - 2Lh + q$$

where

$$b(h,h) = \beta_1 \int_0^T \left[u(0,t;h) - u(0,t;0)\right]^2 dt + \beta_2 \int_0^T \left[u(l,t;h) - u(l,t;0)\right]^2 dt + \alpha \|h\|_H^2,$$

$$Lh = \beta_1 \int_0^T \left[u(0,t;h) - u(0,t;0)\right]\left[y_1(t) - u(0,t;0)\right] dt + \beta_2 \int_0^T \left[u(l,t;h) - u(l,t;0)\right]\left[y_2(t) - u(l,t;0)\right] dt$$

and

$$q = \beta_1 \int_0^T \left[y_1(t) - u(0,t;0)\right]^2 dt + \beta_2 \int_0^T \left[y_2(t) - u(l,t;0)\right]^2 dt.$$

Here $b(h,h)$ is a bilinear, symmetric, continuous and coercive and L is a linear, continuous functional in H.

Then, since the conditions of following theorem given by Lions [18] are hold, the existence and uniqueness of the optimal solution is assured.

Theorem: Let $b(h^*,h)$ continuous symmetric bilinear form with coercive property on H. Then there exists a unique element $h^* \in H$ such that

$$J_\alpha(h^*) = \inf_{h \in H} J_\alpha(h).$$

III. Lagrange Multipliers and Optimality Conditions

To obtain optimality conditions, let us write the augmented functional

$$\tilde{J}_\alpha(h) = \beta_1 \int_0^T \left[u(0,t) - y_1(t)\right]^2 dt + \beta_2 \int_0^T \left[u(l,t) - y_2(t)\right]^2 dt + \alpha \|h\|_H^2 + \int_\Omega \eta\left[p(x)u_{tt} - \left(k(x)u_x\right)_x - F(x,t)\right] dxdt. \quad (12)$$

The Euler equation for this augmented functional will be in the form of

$$\begin{aligned}\tilde{J}'_\alpha(h^*)\delta h &= \int_0^T \left[2\beta_1\left(u^*(0,t) - y_1(t)\right) - k(0)\eta_x(0,t)\right]\delta u^*(0,t) dt \\ &+ \int_0^T \left[2\beta_2\left(u^*(l,t) - y_2(t)\right) + k(l)\eta_x(l,t)\right]\delta u^*(l,t) dt \\ &+ \int_\Omega \left[p(x)\eta_{tt} - \left(k(x)\eta_x\right)_x\right]\delta u^* dxdt \\ &+ \int_0^l p(x)\eta(x,T)\delta u_t^*(x,T) dx - \int_0^l p(x)\eta_t(x,T)\delta u^*(x,T) dx \\ &+ \int_0^T \left[\eta(0,t) + 2\alpha f^*\right]\delta f dt + \int_0^T \left[-\eta(l,t) + 2\alpha g^*\right]\delta g dt.\end{aligned}$$

(13)

Now, we consider the following adjoint boundary-value problem;

$$\begin{aligned} p(x)\eta^*_{tt} - \left(k(x)\eta^*_x\right)_x &= 0 \\ k(0)\eta^*_x(0,t) &= 2\beta_1\left(u^*(0,t) - y_1(t)\right) \\ k(l)\eta^*_x(l,t) &= -2\beta_2\left(u^*(l,t) - y_2(t)\right) \\ \eta^*(x,T) = \eta^*_t(x,T) &= 0. \end{aligned} \quad (14)$$

By the solution of adjoint boundary-value problem, we mean the function $\eta^* \in H^1(\Omega)$ which satisfies the following integral identity;

$$\begin{aligned}\int_0^T \int_0^l \left[p(x)\eta_t^*\phi_t - k(x)\eta_x^*\phi_x\right] dxdt &= 2\beta_1 \int_0^T \left[u^*(0,t) - y_1(t)\right]\phi(0,t) dt \\ &+ 2\beta_2 \int_0^T \left[u^*(l,t) - y_2(t)\right]\phi(l,t) dt - \int_0^l p(x)\eta_t^*(x,0)\phi(x,0) dx\end{aligned} \quad (15)$$

for $\forall\, \phi \in H^1(\Omega)$.

Using the adjoint boundary-value problem the Euler equation becomes

$$\tilde{J}'_\alpha(h^*)\delta h = \int_0^T \left[\eta^*(0,t) + 2\alpha f^*\right]\delta f dt + \int_0^T \left[-\eta^*(l,t) + 2\alpha g^*\right]\delta g dt, \quad \forall \delta h \in H. \quad (16)$$

Theorem: The control $h^* = \{f^*, g^*\}$ and the state

$u^* = u\left(h^*\right)$ are optimal if and only if there exists a multiplier η^* such that h^*, u^* and η^* satisfy the following optimality conditions;

$$\begin{cases} p(x)u^*_{tt} = \left(k(x)u^*_x\right)_x + F(x,t) \\ u^*(x,0) = \varphi_1(x),\ u^*_t(x,0) = \varphi_2(x) \\ k(0)u^*_x(0,t) = f(t),\ \ k(l)u^*_x(l,t) = g(t) \\ p(x)\eta^*_{tt} - \left(k(x)\eta^*_x\right)_x = 0 \\ k(0)\eta^*_x(0,t) = 2\beta_1\left(u^*(0,t) - y_1(t)\right) \\ k(l)\eta^*_x(l,t) = -2\beta_2\left(u^*(l,t) - y_2(t)\right) \\ \eta^*(x,T) = \eta^*_t(x,T) = 0 \\ \eta^*(0,t) + 2\alpha f^* = 0,\ \ -\eta^*(l,t) + 2\alpha g^* = 0 \end{cases}$$

The term $\left\{\eta^*(0,t) + 2\alpha f^*, -\eta^*(l,t) + 2\alpha g^*\right\}$ is called the gradient of $J_\alpha(h)$ at h^* and denoted by the symbol $\nabla J_\alpha\left(h^*\right)$.

IV. Constructing an Iterative Algorithm for Numerical Approximation

In this section, we give an iterative procedure by the method of projection of the gradient to compute a sequence of controls $\{h_k\}$, convergent to the optimal solution h^*.

Select an initial control $h_0 \in H$. If h_k is known $(k \geq 0)$, then h_{k+1} is computed according to the following scheme.

1) Solve the integral equation of (9) in order to obtain the generalized solution u_k.

2) Using u_k, solve the adjoint equation (14) and find η_k.

3) Set

$$h_{k+1} = P_H\left\lfloor h_k - \mu_k \nabla J_\alpha\left(h_k\right)\right\rfloor \tag{17}$$

where $P_H\left\lfloor h_k - \mu_k \nabla J_\alpha\left(h_k\right)\right\rfloor$ is the projection of the element $h_k - \mu_k \nabla J_\alpha\left(h_k\right)$ on the set H.

4) Select the small enough relaxation parameter $\mu_k > 0$ such that

$$J_\alpha\left(h_{k+1}\right) < J_\alpha\left(h_k\right) \tag{18}$$

We refer to the study [12] for the proof of the convergence of the minimizing sequence $\{h_k\}$ to the optimal solution h^*.

V. Conclusion

In the problem of vibration of one dimensional wire, the minimization of the distance between the end points of wire and the desired target functions can be carried out by using the chosen cost functional.

The obtained gradient allows the usage of the method of projection of gradient which converges to the unique optimal solution.

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