Paths on Ordered Edges in Non-oriented Graphs and Economic Networks Modeling

Serge A. Tishchenko and Alexander K. Kovaldji

Abstract—**In many applications, information is naturally represented as networks of interlinked agents. Any communication has its costs and, therefore, the research of best and fastest ways to exchange information is an important field in applied mathematics. Graph theory, economic networks and, more precisely in this article, the study of paths on ordered edges in critical networks can by very fruitful in such modeling problems. In this article we will consider the Gossips and Telephones problem to illustrate such methods in Information Networks modeling.**

Index Terms—**Graph theory, paths on ordered edges, gossip problem.**

I. Introduction

The *Gossip and Telephones problem* is due to A. Boyd and can be formulated as follows: *n* agents at the same time learned *n* different news, and each one of them has learned exactly one news. They begin to call each other to exchange the news. During one call all the news known to callers can be transferred between two respondents. It is impossible to make more than one call at a time. What is the minimal number of phone calls needed so that everyone can know all the news? This problem was solved for the first time and in different way by Brenda Baker and Robert Shostak [1]. In this article we present another solution of this problem based on study of paths on ordered edges in critical graphs.

To start with this problem let us formulate it in mathematical terms of graph theory. The agents will be represented by n vertices of the graph and the calls between agents will be represented by the edges connecting the corresponding vertices. According to the problem, all calls can be numbered in chronological order, we will keep this order in the graph. Let us suppose that during the calls the agents have exchanged all the news. In this case, the news of an agent will be known to another only if the corresponding oriented pair of vertices is connected by a path, in which numbers of edges form an increasing sequence. The condition of the problem is satisfied if such a path connects each oriented pair of vertices in the graph. In more formal language of graph theory all of the above can be wrote as follows:

Let *G* a finite connected graph without loops with a given numeration of its edges $N_G(E)$. We call the numeration *correct*, if every oriented pair of vertices $(v_1, v_2) \in V(G)$ is connected by *increasing path* $R_G^+(v_1, v_2)$, the numbers of edges in which form an increasing sequence. Obviously, the pair (v_1, v_2) is also connected by a decreasing path $R^-(v_1, v_2) = \overline{R^+(v_2, v_1)}$, the numbers of edges in which form a decreasing sequence. Let us call a graph *correct*, or a c-graph if there is a correct numbering of its edges.

We are interested in the minimum number of calls-edges *m* for a given number of agents $(n \geq 2)$. Therefore, the solution of the problem comes down to the finding of the function $m(n)$. We plan to prove the following theorem:

Theorem.

$$m(n) = \begin{cases} 2n-3, & 2 \leq n < 4, \\ 2n-4, & n \geq 4. \end{cases} \tag{1}$$

II. Preliminary Results

Before we can look for the best solution, it is necessary to prove that the solution exists. To prove this we will use the mathematical induction proof technique:

Lemma 1. $\forall\, n \in \mathrm{N},\, n \geq 2$ the set of c-graphs is not empty. Then $n = 2$ it is enough to connect the two vertices by a single edge. Let the number of this edge be 1. Thus (fig. 1a), the resulting graph is a c-graph. Let us assume that then $n = k,\, k \geq 2$ there exist a c-graph *G* with *k* vertices. Let us consider its correct numbering $N_G(E)$ and let us increase the number of each vertex by one. We add to *G* a vertex *v*, connecting it with an arbitrary vertex $v' \in V(G)$ by two edges (fig. 1b). One of these edges will have the number 1 and the other will have the number equal to $\max\{N_G(E)\}+1$. Obviously, obtained numeration $N'_{G'}(E')$ of the graph G' is correct, thus, the set of c-graphs of size $k+1$ is not empty. The lemma is proved.

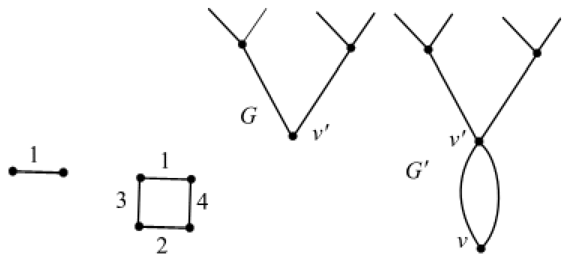

Fig. 1a. c-graphs of size 2 and 4. Fig. 1b. The construction of a *c*-graph of size $k+1$.

The set of c-graphs of size *n* is not empty. Thus, there exist

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a c-graph G with a minimal number of edges $|E(G)| = \min_{|V(G)|=n}$ (let us call it *minimal*, or a *mc-graph*). The function $m(n)$ is defined:

$$m(n) = \min_{|V(G)|=n} |E(G)|. \quad (2)$$

We have proved the existence of solutions for all $n \geq 2$. Now, using the construction from Lemma 1, we obtain an upper bound for the function $m(n)$ increasing n by one:

Lemma 2.

$$\forall\, n \in \mathrm{N},\, n \geq 2,\ m(n+1) \leq m(n) + 2 \quad (3)$$

Let us consider a mc-graph G of size n with number of edges $|E(G)| = m(n)$ and an arbitrary correct numeration $N_G(E)$. We increase the numbers of all edges by one. We add to the graph G a vertex v, such that it is connected to $v' \in V(G)$ with two edges (fig. 1b). Let one of these edges have the number 1 and the other will have number greater than $\max\{N_G(E)\}+1$. The new numeration $N'_{G'}(E')$ of the new graph G' is correct, G' is a c-graph of size $n+1$ and we have $m(n+1) \leq m(n) + 2$.

Examples of c-graphs of size 2 and 4 on the Figure 1a, together with Lemma 2 proves the lower bound of the function $m(n)$ to be:

Corollary 2.1.

$$m(n) \leq \begin{cases} 2n-3, & 2 \leq n < 4, \\ 2n-4, & n \geq 4. \end{cases} \quad (4)$$

So we found a way to transfer all of the news making $2n-4$ calls for $n \geq 4$ ($2n-3$ calls for $n = 2$ and $n = 3$). Now we have to prove that this solution is the best in terms of number of calls. To this end we will need the following well-known result:

Lemma 3. Number of edges in G of size n consisting of p connected components is:

$$|E(G)| \geq n - p \quad (5)$$

When $|E(G)| = n - p$ the graph G is a forest containing p trees.

III. Particular Cases

Let us consider the case $2 \leq n \leq 4$.

Theorem (case** $2 \leq n \leq 4$**). $m(2) = 1$, $m(3) = 3$, $m(4) = 4$.

Let us consider a connected c-graph G (each pair of vertices is connected by a path). Therefore $p = 1$, $m(n) \geq n - 1$ and $m(2) = 1$. In the case of $n = 3$ let us suppose by contradiction $m(3) = 2$ (in the case of $n = 4$ let us suppose $m(4) = 3$). The c-graph G is a tree (each pair of vertices is connected only by a unique path). Increasing paths in both directions pass through the same edges. This is possible only when the path consists of only one edge (its length is 1). Each pair of vertices is connected by an edge, $m(3) \geq 3$ (and if $n = 4$, $m(4) \geq 6$). This is contrary to the initial hypothesis.

Note that the first result is quite obvious, as the two friends must have at least one call to share the news. To prove the other two results, we have used graph theory.

IV. Key Lemmas

If we assume that the equality $m(n) = m(n-1) + 2$ is correct for all $n \geq 5$, the Theorem will be proved. Thus, on the contrary, we are interested by the *critical* value of the parameter n_c, such that for the first time from $n = 5$

$$m(n_c) < m(n_c - 1) + 2. \quad (6)$$

We call G_c the *critical* mc-graph of size n_c. Let the number of incident edges of the vertex $v \in V(G)$ be denoted by $d_G(v)$. Let us now prove the following property of such graph:

Lemma 4. In the critical mc-graph G_c there is no cyclic ascending path $R^+_{G_c}(v,v)$, $v \in V(G_c)$.

Ad absurdum, we consider $v \in V(G_c)$ and a closed ascending path $R^+_{G_c}(v,v)$ (fig. 2). The first e_1 and the last e_2 edges of that path are incident to v, and $N_{G_c}(e_1) < N_{G_c}(e_2)$. We have, $e_1 \neq e_2$ and $d_{G_c}(v) \geq 2$.

In the case of $d_{G_c}(v) = 2$ let us consider the graph $G' = G_c \setminus \{v, e_1, e_2\}$, with the same edge numeration $N_{G_c}(E)$. Obviously, any pair of vertices $(v_0, v'_0) \in G'$, connected in the graph G_c by an ascending path $R^+_{G_c}(v_0, v'_0)$, passing through the pair of edges (e_1, e_2) (Fig. 2), is connected in the G' by another ascending path $R^+_{G'}(v_0, v'_0)$, such that it is obtained from $R^+_{G_c}(v_0, v'_0)$ by the replacement of the segment v_1, e_1, v_c, e_2, v_2 with the ascending path $R^+_{G'}(v_1, v_2) = R^+_{G_c}(v,v) \setminus e_1, v, e_2$. Therefore, $d_{G_c}(v) > 2$ because otherwise the graph G' would be a c-graph and we will have $m(n_c) \geq m(n_c - 1) + 2$, what is impossible.

We now construct the new graph G'. For this end we will use $G'' = G_c \setminus \{v, e_1, e_2\}$, with the same edge numeration $N_{G_c}(E)$. For all edges from v: e_i, $3 \leq i \leq d(v)$, we will follow the next procedure: in the path $R'^+_{G_c}(v_1, v_2) = R^+_{G_c}(v,v) \setminus e_1, v, e_2$ we choose two consecutive edges e' and e'', such that $N_{G_c}(e') < N_{G_c}(e_i) < N_{G_c}(e'')$, and a vertex v'_i from $R'^+_{G_c}(v_1, v_2)$ between these two edges (Fig. 2). In the case then the number $N_{G_c}(e_i)$ is smaller than the number of the first edge in the path $R'^+_{G_c}(v_1, v_2)$, we will choose $v'_i = v_1$, or else if $N_{G_c}(e_i)$ is greater than the number of the last edge in $R'^+_{G_c}(v_1, v_2)$, we will choose $v'_i = v_2$. Let us consider now the second vertex v_i, incident to the edge e_i.

If $v_i' \neq v_i$, we will add an edge e_i' between the vertices v_i and v_i', and we will give it the number of the edge e_i. And we delete the edge e_i.

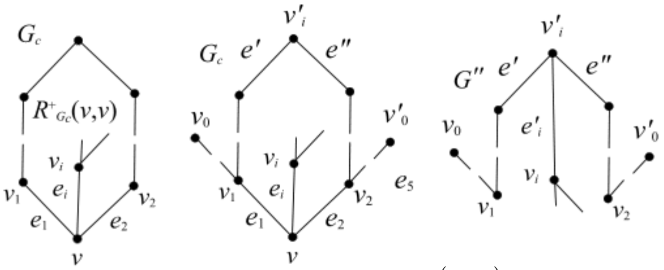

Fig. 2. Cyclic ascending path $R^+_{G_c}(v\ ,v\ )$.

Let us consider the graph $G' = G'' + \{e_i', i = 3,.., d(v)\}$ and an ascending path $R^+_{G_c}(v_0, v_0')$, passing through the pair of edges (e_i, e_j), $i \geq 3$, $j = 1,2$. Without loose of generality let us assume $N_{G_c}(e_i) < N_{G_c}(e_j)$ (the case $N_{G_c}(e_i) > N_{G_c}(e_j)$ is treated similarly). The vertex v_i' separate the path $R^+_{G'}(v_1, v_2)$ into two ascending paths $R^+_{G'}(v_1, v_i')$ and $R^+_{G'}(v_i', v_2)$ (one of them can be of length 0). Obviously, any pair of vertices $(v_0, v_0') \in G'$, connected in the graph G_c by an ascending path $R^+_{G_c}(v_0, v_0')$, passing though the pair of edges (e_i, e_j) (Fig. 2), is connected in G' by another ascending path $R^+_{G'}(v_0, v_0')$, obtained from $R^+_{G_c}(v_0, v_0')$ by the replacement of the segment e_i, v, e_j by the segment $e_i', R_{G'}(v_i', v_j)$.

For the case $d_{G_c}(v) > 3$ we also consider an ascending path $R^+_{G_c}(v_0, v_0')$, passing through the pair of edges $(e_i, e_{i'})$, $i, i' \geq 3$. Without loose of generality let us assume $N_{G_c}(e_i) < N_{G_c}(e_{i'})$ (the case $N_{G_c}(e_i) > N_{G_c}(e_{i'})$ is treated similarly). The vertex $v_{i'}'$ separate the path $R^+_{G'}(v_i', v_2)$ into two ascending paths $R^+_{G'}(v_i', v_{i'}')$ and $R^+_{G'}(v_{i'}', v_2)$ (each one of them can be of length 0). Obviously, any pair of vertices $(v_0, v_0') \in G'$, connected in the graph G_c by an ascending path $R^+_{G_c}(v_0, v_0')$, passing though the pair of edges $(e_i, e_{i'})$, is connected in G' by another ascending path $R^+_{G'}(v_0, v_0')$, obtained from $R^+_{G_c}(v_0, v_0')$ by the replacement of the segment $e_i, v, e_{i'}$ by the segment $e_i', R_{G'}(v_i', v_{i'}'), e_{i'}'$.

In the particular case of $v_i' = v_i$ everything is as in the general case, except of the edge e_i', that we shall not need and that edge can be deleted.

The set of all ascending paths in G' contains also all the ascending paths from G''. We can conclude that G' is a c-graph and $m(n_c) \geq m(n_c - 1) + 2$, what is impossible.

Lemma is proved.

Corollary 4.1.

1) In critical graph, if an edge $e \in E(G_c)$ incident to vertices $v, v' \in V(G_c)$ has a maximal number between all edges incident to the vertex v than it also has the maximal number between all the edges incident to v';

2) In critical graph, if an edge $e \in E(G_c)$ incident to vertices $v, v' \in V(G_c)$ has a minimal number between all edges incident to the vertex v than it also has the minimal number between all the edges incident to v'.

It suffices to prove that in critical graph, if an edge $e \in E(G_c)$ incident to vertices $v, v' \in V(G_c)$, has a maximal number between all edges incident to the vertex v than it also has the maximal number between all the edges incident to v' (the case of the minimal number is proved by analogy). Ad absurdum, let us consider an edge $e' \in E(G_c)$, incident to the vertices $v', v'' \in V(G_c)$ such that $N_{G_c}(e') > N_{G_c}(e)$. If the ascending path $R^+_{G_c}(v'', v)$ passes though the edge e than there exist the cyclic ascending path $R^+_{G_c}(v'', v'') = [R^+_{G_c}(v'', v) \setminus e, v], e', v''$. In the other case, there exist the cyclic ascending path $R^+_{G_c}(v'', v'') = R^+_{G_c}(v'', v), e, v', e', v''$. In both cases we have a contradiction with the Lemma 4.

We denote by $\widetilde{E}(G)$ the set of all edges in a c-graph G such that these edges have nor the minimal nor the maximal number between all the edges incident to any of the two vertices these edges are incident to.

Corollary 4.2. In a critical graph each pair of vertices $(v_1, v_2) \in V(G_c)$ is connected by an ascending path $R^+_{G_c}(v_1, v_2)$ with all edges except maybe the first one and the last one of this path are in $\widetilde{E}(G_c)$.

Let us consider a vertex $v_c \in V(G_c)$ in the mc-graph. We construct the graph $G^+_{v_c}$ that is the union of all ascending paths $R^+_G(v_c, v_i)$ and the graph $G^-_{v_c}$ that is the union of all descending paths $R^-_G(v_c, v_i)$ that connects the vertex v_c with all the other vertices v_i of the graph G.

Lemma 5. The intersection $E(G^+_{v_c}) \cap E(G^-_{v_c})$ consists of edges that are incident to the vertex v_c.

Ad absurdum, we suppose, that the intersection $E(G^+_{v_c}) \cap E(G^-_{v_c})$ has an edge e, that is not incident to v_c. Thus there exist an ascending path $R^+_{G_c}(v_c, v')$ and a descending path $R^-_{G_c}(v_c, v'')$ passing through the edge e. Let us consider the parts of these paths before they enter the e: $R^+_{G_c}(v_c, v_1)$ and $R^-_{G_c}(v_c, v_2)$ there v_1 and v_2 are the vertices incident to the edge e (Fig. 3). If the edge e in both paths $R^+_{G_c}(v_c, v')$ and $R^-_{G_c}(v_c, v'')$ is crossed in opposing directions than the cyclic path $R_{G_c}(v_c, v_c) = R^+_{G_c}(v_c, v_1), e, \overline{R^-_{G_c}(v_c, v_2)}$ is an ascending one and this is in contradiction with the Lemma 4. Otherwise if the edge e in the paths $R^+_{G_c}(v_c, v')$ and $R^-_{G_c}(v_c, v'')$ is crossed in the same direction than the cyclic path $R_{G_c}(v_c, v_c) = R^+_{G_c}(v_c, v_1) \setminus v_1, \overline{R^-_{G_c}(v_c, v_1)}$ is an ascending one and this is in contradiction with the Lemma 4. The lemma is proved.

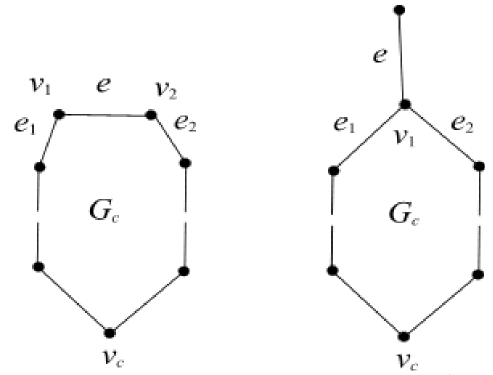

Fig. 3. Construction of the cyclic ascending path $R^+_{G_c}(v_c, v_c)$.

Let $\delta(G) = \min_{v \in V(G)} \{d_G(v)\}$. We will call *critical* the vertex v_c such that $d(v_c) = \delta(G)$.

Corollary 5.1. $\delta(G_c) \geq 3$. (7)

Let us take an arbitrary critical vertex $v_c \in V(G_c)$. The graphs $G^+_{v_c}$ and $G^-_{v_c}$ are connected spinning subgraphs of the graph G_c. Therefore

$$\left|E\left(G^+_{v_c}\right)\right| \geq n_c - 1, \ \left|E\left(G^-_{v_c}\right)\right| \geq n_c - 1. \tag{8}$$

Using the Lemma 5 we have,

$$\left|E\left(G^+_{v_c}\right) \cap E\left(G^-_{v_c}\right)\right| \leq \delta(G_c). \tag{9}$$

Using (1), (8) and (9), we have

$$\begin{aligned} m(n_c) = |E(G_c)| \geq \left|E\left(G^+_{v_c}\right) \cup E\left(G^-_{v_c}\right)\right| = \left|E\left(G^+_{v_c}\right)\right| \\ + \left|E\left(G^-_{v_c}\right)\right| - \left|E\left(G^+_{v_c}\right) \cap E\left(G^-_{v_c}\right)\right| \geq 2n_c - 2 - \delta(G_c) \end{aligned} \tag{10}$$

Bringing (4) together with (6) we obtain $m(n_c) < m(n_c - 1) + 2 \leq 2n_c - 4$ and therefore

$$m(n_c) \leq 2n_c - 5 \tag{11}$$

Together with (10) we obtain (7).

Corollary 5.2. For any critical vertex $v_c \in V(G_c)$

$$E\left(G^+_{v_c}\right) \cup E\left(G^-_{v_c}\right) = E(G_c) \tag{12}$$

Comparing the obvious inequality $2m(n_c) \geq n_c \delta(G_c)$ with (11) we have:

$$2n_c - 2 - \delta(G_c) \geq 2n_c - 2 - \frac{2(2n_c - 5)}{n_c} > 2n_c - 6 \tag{13}$$

Therefore $2n_c - 2 - \delta(G_c) \geq 2n_c - 5$ and together with (11) this is possible only if all the inequalities in (10) are the exact equalities.

Let us consider an arbitrary critical vertex $v_c \in V(G_c)$ and the graphs $G^+_{v_c}$ and $G^-_{v_c}$. Let the edge e_1 have the minimal number between all edges incident to the vertex v_c. Let v_1 be the second vertex incident to e_1. Let the edge e_2 have the maximal number between all edges incident to the vertex v_c. Let v_2 be the second vertex incident to e_2.

Lemma 6. Each vertex $v \in V(G_c) \setminus \{v_c, v_1, v_2\}$ is connected to at least one of the vertices v_c, v_1, v_2 by a path such that all its edges are in $\tilde{E}(G_c)$.

In virtue of (15) and the Corollary 2.2, any vertex v is incident to at least on edge $e \in \tilde{E}(G_c)$. According to (16) there exist either an ascending $R^+_{G_c}(v_c, \ldots)$ either a descending $R^-_{G_c}(v_c, \ldots)$ path crossing the edge e. All edges in such path before the edge e except maybe the first one are in $\tilde{E}(G_c)$ due to the Corollary 4.2. Therefore, the vertex v is connected to at least one of the vertices v_c, v_1, v_2 by a path such that all its edges are in $\tilde{E}(G_c)$.

V. Theorem

Let us summarize our analysis. Assuming the existence of a critical mc-graph, we have established a number of its important properties (Lemma 6, Corollaries 5.1 and 5.2). Now we are ready for the proof of the main theorem based on mutual inconsistency of these properties.

Theorem. (Proof)

In accordance with Lemma 6, the spanning graph $\tilde{G} = G_c \setminus \left[E(G_c) \setminus \tilde{E}(G_c)\right]$, obtained from graph G_c by the deletion of all edges do not belonging to $\tilde{E}(G_c)$, consists of no more than three connected components. Therefore, in accordance with Lemma 3,

$$\left|\tilde{E}(G_c)\right| = \left|E(\tilde{G})\right| \geq n_c - 3. \tag{14}$$

By virtue of (7), at least two edges incident to an arbitrary vertex $v \in G_c$, are not in $\tilde{E}(G_c)$. Therefore, considering (11),

$$\left|\tilde{E}(G_c)\right| \leq m(n_c) - n_c \leq n_c - 5, \tag{15}$$

which contradicts (14). The assumption of the existence of a critical graph was wrong and thus the Theorem is proved.

Let's go back to the original problem. In accordance with the Theorem, $2n - 4$ calls are necessary for four or more agents and $2n - 3$ calls are necessary in the case of $2 \leq n \leq 3$.

VI. Conclusion

In this article we have presented a combinatorial approach based on path constructions in a network with fixed link numbering. As we show in the case of gossip problem example, this approach becomes effective then studying and proofing optimal solutions produced by any model working with non-oriented networks of agents. More information on gossip problem can be found in the papers of Krumme and

Bumby [2], [3]. Label-connected graphs and information propagation on graphs is discussed in [4]-[6]. Communication problem for graphs and digraphs is described in [7]. Research of minimal time and further gossip problems are studied in [8]-[10]. More information on economic networks and graph modeling for agent networks can be found in the lecture notes by Konig and Battiston in [11].

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