Some Properties of Soft Categories

Simge Öztunç

Abstract—In this paper we recalled some definitions and properties of soft set theory and soft category theory. We investigated new properties of soft category theory such as concept of initial and terminal objects. Also we give the definition of soft connected category and we obtain the necessary and sufficient condition to have a zero object for soft category.

Index Terms—Soft category, initial object, terminal object, connected soft category.

I. INTRODUCTION

The concept of soft sets was introduced by D. Molodtsov [1] in 1999 and Soft set theory became an alternative and useful tool for computer science, modeling problems in engineering, economics, medical and social science.

Theoretical properties of soft set theory has also been studied some mathematicians. Maji *et al.* [2] introduced several operations on soft sets. Aktaş and Çağman [3] defined the notion of soft groups and developed the algebraic properties of soft groups. Some concepts in soft topological spaces studied by Zorlutuna *et al.* [4] Also Aras, Sönmez, Çakallı [5] and Zorlutuna and Çakır [6] worked on continuity of soft mappings.

On the other hand Category Theory was studied by many mathematicians. Eilenberg and Mac Lane's 'General Theory of Naturel Equivalences' [7] was the first original paper in which the theory was formulated. Category Theory has an important place, because it plays multidisciplinary roles among many branches of mathematics. Also Category Theory may be considered as study of (abstract) algebras of functions.

The soft category theory studied by Sardar and Gupta in [8]. They introduced the basic notions of the theory of soft categories and gave some introductory result of the soft category theory.

The purpose of this paper is to study some new properties of soft category theory. Sardar and Gupta defined in [8] the initial and terminal objects and zero objects in soft categories and we show that objects of the same type in a soft category are isomorphic. Also we defined connected soft category and gave the necessary and sufficient condition to be connected soft category.

Manuscript received April 24, 2015; revised April 23, 2016. This Paper is supported by Celal Bayar University Scientific Research Project Coordination Unit (CBU BAP- Project No:2015-109).

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II. PRELIMINARIES

In this section we give some definitions of Soft Set Theory and Category Theory.

Definition 2.1. [2] A pair (F, A) is called a soft set over the universe X , where F is a mapping given by

$$F : A \rightarrow P(X)$$

and $A \subseteq E$.

Any soft set (F, A) can be extended to a soft set of type (F, E) , where

$$F(e) \neq \emptyset$$

for all $e \in A$ and

$$F(e) = \emptyset$$

for all $e \in E \setminus A$. $S(X, E)$ denotes the family of all soft sets over X [2].

Definition 2.2. [2, 9] Let $(F, A), (G, B) \in S(X, E)$.

1) (F, A) is a soft subset of (G, B) , denoted by $(F, A) \subseteq (G, B)$,

if $F(e) \subseteq G(e)$ for each $e \in E$.

2) (F, A) and (G, B) are said to be soft equal, denoted by

$$(F, A) = (G, B),$$

if $(F, A) \subseteq (G, B)$ and $(G, B) \subseteq (F, A)$.

3) Union of (F, A) and (G, B) is a soft set (H, C) , where

$$C = A \cup B \text{ and } H(e) = F(e) \cup G(e)$$

for each $e \in E$. This is denoted by

$$(F, A) \cup (G, B) = (H, C)$$

4) Intersection of (F, A) and (G, B) is a soft set (H, C) ,

where

$$C = A \cap B \text{ and } H(e) = F(e) \cap G(e)$$

for each $e \in E$. This relationship is written as

$$(F, A) \tilde{\cap} (G, B) = (H, C).$$

Definition 2.3. [2], [9] Let $(F, A) \in S(X, E)$. (F, A) is said to be a null soft set, denoted by Φ , if $\forall e \in E, F(e) = \emptyset$.

Definition 2.4. [2] A soft set (F, A) over X is said to be an absolute soft set, denoted by $\tilde{A}$ if for all $a \in A, F(a) = X$.

Definition 2.5 [10] A category $\mathcal{C}$ consists of the following data:

- Objects: $A, B, C, \dots$
- Arrows: $f, g, h, \dots$
- For each arrow f , there are given objects $dom(f), cod(f)$

called the domain and codomain of f . We write

$$f : A \rightarrow B$$

To indicate that

$$A = dom(f) \text{ and } B = cod(f)$$

Given arrows $f : A \rightarrow B$ and $g : B \rightarrow C$, that is, with

$$cod(f) = dom(g)$$

there is given an arrow

$$g \circ f : A \rightarrow C$$

called the composite of f and g .

For each object A , there is given an arrow

$$1_A : A \rightarrow A$$

called the identity arrow of A .

This data required to satisfy the following laws:
Associativity :

$$h \circ (g \circ f) = (h \circ g) \circ f$$

for all $f : A \rightarrow B, g : B \rightarrow C$ and $h : C \rightarrow D$.

Unit :

$$f \circ 1_A = f = 1_B \circ f$$

for all $f : A \rightarrow B$.

Definition 2.6. [10] In any category $\mathcal{C}$, an arrow

$f : A \rightarrow B$ is called an monomorphism, if given any $g, h : C \rightarrow A, f \circ g = f \circ h$ implies $g = h$,

$$C \xrightarrow[h]{g} A \xrightarrow{f} B$$

Definition 2.7. [10] In any category $\mathcal{C}$, an arrow $f : A \rightarrow B$ is called an monomorphism, if given any $i, j : B \rightarrow D, i \circ f = j \circ f$ implies $i = j$,

$$A \xrightarrow{f} B \xrightarrow[i]{j} D$$

Definition 2.8. [10] In any category $\mathcal{C}$, an arrow $f : A \rightarrow B$ is called an isomorphism, if there is an arrow

$$g : B \rightarrow A$$

in $\mathcal{C}$ such that

$$g \circ f = 1_A \text{ and } f \circ g = 1_B.$$

Definition 2.9. [10] A functor

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

Between $\mathcal{C}$ and $\mathcal{D}$ is a mapping of objects to objects and arrows to arrows in such a way that

- $F(f : A \rightarrow B) : F(f) : F(A) \rightarrow F(B)$
- $F(1_A) : 1_{F(A)}$
- $F(g \circ f) = F(g) \circ F(f)$.

That is F preserves domains and codomains, identity arrows, and composition. A functor

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

Thus gives a sort of picture of $\mathcal{C}$ in $\mathcal{D}$.

Definition 2.10. [11] An object U of a category $\mathcal{C}$ is said to be an initial object if for every object X of $\mathcal{C}$, the set $Mor_{\mathcal{C}}(U, X)$ is a singleton. Dually U is said to be a terminal object if for every object X of $\mathcal{C}$, the set

$$Mor_{\mathcal{C}}(X, U)$$

is a singleton

III. SOME PROPERTIES OF SOFT CATEGORIES

Definition 3.1. [8] Let $\mathcal{C}$ be a category, $\mathcal{P}(\mathcal{C})$ is the set of all subcategories of $\mathcal{C}$ and A a set of parameters. Let $F : A \rightarrow \mathcal{P}(\mathcal{C})$ be a mapping. Then (F, A) is said to be a soft category over $\mathcal{C}$ if $F(x)$ is a subcategory of $\mathcal{C}$.

Definition 3.2. [8] Let (F, A) be a soft category over $\mathcal{C}$. Then

$$(F, A)$$

is said to be a soft category with initial objects over $\mathcal{C}$ if $F(x)$, as a category has initial objects for all $x \in A$.

Definition 3.3. [8] Let (F, A) be a soft category over $\mathcal{C}$. Then

$$(F, A)$$

is said to be a soft category with terminal objects over $\mathcal{C}$ if $F(x)$, as a category has terminal objects for all $x \in A$.

Theorem 3.4. Let (F, A) be a soft category over $\mathcal{C}$. Then any two initial objects in a soft category (F, A) are isomorphic.

Proof: Assume that (F, A) is a soft category over $\mathcal{C}$. Let U_1 and U_2 be initial objects in a soft category (F, A) . Then U_1 and U_2 are initial objects in $F(x)$. We must show contains an isomorphism. If U_1 is an initial object for $F(x)$, then

$$\begin{aligned} Mor_{F(x)}(U_1, U_2) \\ Mor_{F(x)}(U_1, U_2) = \{\alpha\} \end{aligned}$$

For any object U_2 . If U_2 is an initial object for $F(x)$, then

$$Mor_{F(x)}(U_2, U_1) = \{\beta\}$$

For any object U_1 . We get

$$\alpha \circ \beta \in Mor_{F(x)}(U_2, U_2)$$

By the composition property of morphisms. Therefore $\alpha \circ \beta = id_{U_2}$.

Similarly we have

$$\beta \circ \alpha = id_{U_1}.$$

It follows that α is an isomorphism with

$$\beta = \alpha^{-1}$$

and U_1 and U_2 are isomorphic.

Theorem 3.5. Let (F, A) be a soft category over $\mathcal{C}$ and $f \in Mor_{F(x)}(X, A)$. If X is a terminal object in a soft category (F, A) . Then, f is monomorphism.

Proof: Let (F, A) be a soft category over $\mathcal{C}$. Assume that X is a terminal object and in $F(x)$ and $h, k \in Mor_{F(x)}(C, X)$. Let $f \circ h = f \circ k$. Consider the following diagram

$$C \xrightarrow{h, k} X \xrightarrow{f} A$$

We obtained $h = k$, since X is a terminal object in $F(x)$. Hence f is a monomorphism.

Theorem 3.6. Let (F, A) be a soft category over $\mathcal{C}$. Then any two terminal objects in a soft category (F, A) are isomorphic.

Proof: Let (F, A) be a soft category over $\mathcal{C}$. Assume that O_1 and O_2 are terminal objects in a soft category (F, A) . Then O_1 and O_2 are terminal objects in $F(x)$. We have to show that

$$Mor_{F(x)}(O_1, O_2)$$

Contains an isomorphism. If O_1 is a terminal object for $F(x)$, then

$$Mor_{F(x)}(O_2, O_1) = \{f\}$$

For any object O_2 . If O_2 is a terminal object for $F(x)$, then

$$Mor_{F(x)}(O_1, O_2) = \{g\}$$

For any object O_1 . We have

$$g \circ f = id_{O_1}$$

And similarly

$$f \circ g = id_{O_2}$$

By the composition property of morphisms. Therefore f is an isomorphism with

$$f = g^{-1}$$

And O_1 and O_2 are isomorphic.

Corollary 3.7. The objects of the same type in a soft category are isomorphic.

Proof: The proof is obtained by Theorem 3.5 and Theorem 3.6.

Definition 3.8. [8] Let (F, A) be a soft category over $\mathcal{C}$. Then

$$(F, A)$$

is said to be a soft category with zero objects over $\mathcal{C}$ if $F(x)$, as a category has zero object.

Definition 3.9. A soft category over $\mathcal{C}$ is said to be connected if

$$Mor_{F(x)}(A, B) \neq \emptyset$$

for all objects A, B of (F, A) .

Theorem 3.10. Let (F, A) be a soft category over a category $\mathcal{C}$ with an initial object and a terminal object. Then (F, A) has a zero object if and only if (F, A) is connected.

Proof: If Z is a zero object for (F, A) . Hence Z a zero object for $F(x)$. Then for every morphism α , we have

$$\begin{aligned} X &\xrightarrow{\alpha} A \xrightarrow{0_{AB}} B = X \xrightarrow{\alpha} A \xrightarrow{0_{AZ}} Z \xrightarrow{0_{ZB}} B \\ &= X \xrightarrow{0_{XZ}} Z \xrightarrow{0_{ZB}} B \\ &= X \xrightarrow{0_{XB}} B \end{aligned}$$

For any object X, A, B of soft category (F, A) . Writing all zero morphisms as 0 , It follows that $0 \circ \alpha = 0$ and similarly we have $\alpha \circ 0 = 0$. Hence it is obtained that (F, A) is connected.

Conversely suppose that (F, A) is connected. Let X be an initial object and Y is a terminal object for $F(x)$. We observe that if

$$Mor_{F(x)}(X, Y) = \{\alpha\} \neq \emptyset,$$

Then α is monic. In fact since (F, A) is connected. There exists

$$\beta: Y \rightarrow X$$

and we have

$$\beta \circ \alpha = id_X$$

Since X is initial. Thus α is a section and hence is monic. Assume that

$$f, g \in Mor_{F(x)}(A, X).$$

Then

$$\alpha \circ f, \alpha \circ g \in Mor_{F(x)}(A, Y)$$

From the composition property. We get

$$\alpha \circ f = \alpha \circ g.$$

Since α is monic, it is obtained that

$$f = g.$$

Consequently we see that X is also terminal object for $F(x)$, hence is a zero object.

IV. CONCLUSION

We have worked on some properties of soft categories in this paper. We concluded that the objects of the same type in a soft category are isomorphic. Also we defined the connectedness of a soft category and consequently obtained the sufficient and necessary condition to be soft connected.

ACKNOWLEDGMENT

Author(s) thanks to Celal Bayar University Scientific Research Project Coordination Unit for their financial support.

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