Divergence and Curl Operators in Skew Coordinates

Rajai S. Alassar and Mohammed A. Abushoshah

***Abstract*—The divergence and curl operators appear in numerous differential equations governing engineering and physics problems. These operators, whose forms are well known in general orthogonal coordinates systems, assume different casts in different systems. In certain instances, one needs to custom-make a coordinates system that my turn out to be skew (i.e. not orthogonal). Of course, the known formulas for the divergence and curl operators in orthogonal coordinates are not useful in such cases, and one needs to derive their counterparts in skew systems. In this note, we derive two formulas for the divergence and curl operators in a general coordinates system, whether orthogonal or not. These formulas generalize the well known and widely used relations for orthogonal coordinates systems. In the process, we define an orthogonality indicator whose value ranges between zero and unity.**

***Index Terms*—Coordinates systems, curl, divergence, Laplace, skew systems.**

I. Introduction

The curl and divergence operators play significant roles in physical relations. They arise in fluid mechanics, elasticity theory and are fundamental in the theory of electromagnetism, [1], [2].

The physical significance of the Curl of a vector field $\vec{F}$, denoted by $\vec{\nabla}\times\vec{F}$, is that it measures the amount of rotation or angular momentum of the contents of a given region of space. If the value of the curl is zero then the field is said to be irrotational. The curl is defined in an arbitrary *orthogonal* curvilinear coordinates (u_1,u_2,u_3) as

$$\vec{\nabla}\times\vec{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{e_1} & h_2\boldsymbol{e_2} & h_3\boldsymbol{e_3} \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ F_1h_1 & F_2h_2 & F_3h_3 \end{vmatrix} \tag{1}$$

where $\boldsymbol{e}_i$ is a unit vector in the direction of u_i, and $\vec{F}=F_1\boldsymbol{e_1}+F_2\boldsymbol{e}_2+F_3\boldsymbol{e_3}$. The length of the tangent vector in the direction of u_i is known as scale factor, h_i, and is defined by

$$h_i=\left|\frac{\partial\vec{r}}{\partial u_i}\right| \tag{2}$$

where $\vec{r}$ is the position vector in any three dimensional space, i.e. $\vec{r}=\vec{r}(u_1,u_2,u_3)$.

Note that $\dfrac{\partial\vec{r}}{\partial u_i}$ is a tangent vector to the u_i curve where the other two coordinates variables remain constant. A unit tangent vector in this direction, therefore, is

$$\boldsymbol{e_i}=\frac{\dfrac{\partial\vec{r}}{\partial u_i}}{\left|\dfrac{\partial\vec{r}}{\partial u_i}\right|}=\frac{\dfrac{\partial\vec{r}}{\partial u_i}}{h_i}\quad\text{or}\quad\frac{\partial\vec{r}}{\partial u_i}=h_i\boldsymbol{e_i} \tag{3}$$

The Divergence of a vector field over a control volume V bounded by the surface S, denoted by $\vec{\nabla}\bullet\vec{F}$, is defined by

$$\vec{\nabla}\bullet\vec{F}=\lim_{V\to 0}\frac{\oint_S \vec{F}\bullet\vec{n}\,dS}{V} \tag{4}$$

where $\vec{n}$ is an outward unit normal vector to the surface S. The divergence actually measures the net outflow of a vector field from an infinitesimal volume around a given point (or how much a vector field "converges to" or "diverges from" a given point). The general expression of the divergence for arbitrary *orthogonal* curvilinear coordinates is given by

$$\vec{\nabla}\bullet\vec{F}=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(F_1h_2h_3)+\frac{\partial}{\partial u_2}(F_2h_1h_3)+\frac{\partial}{\partial u_3}(F_3h_1h_2)\right] \tag{5}$$

The Laplacian operator of a scalar $\psi=\psi(u_1,u_2,u_3)$, denoted by $\nabla^2\psi$ is defined as the divergence of the gradient of ψ; that is

$$\nabla^2\psi=\vec{\nabla}\bullet(\vec{\nabla}\psi)=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right] \tag{6}$$

where

$$\vec{\nabla}\psi=\frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\boldsymbol{e_1}+\frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\boldsymbol{e_2}+\frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\boldsymbol{e_3} \tag{7}$$

Equations (1) and (5) are valid for *orthogonal* systems

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The authors are with the Department of Math and Stat, King Fahd University of Petroleum & Minerals, Dhahran, Saudi Arabia (e-mail: alassar@kfupm.edu.sa.sa, abushosha@kfupm.edu.sa).

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only (i.e. $\boldsymbol{e_1}$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ are orthogonal), [3]-[5].

II. Divergence for General Coordinates Systems

The Divergence Theorem relates the flow, or the flux, of a vector field through a surface to the behavior of the vector field inside the surface. More precisely, it states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface, i.e.

$$\iint_S \vec{w}\bullet\vec{n}\, dS = \iiint_D \vec{\nabla}\bullet\vec{w}\, dv \tag{8}$$

where D is a closed bounded region with piecewise smooth boundary S, $\vec{n}$ is an outer unit vector normal to the surface S, $\vec{w} = w_1\boldsymbol{e_1} + w_2\boldsymbol{e_2} + w_3\boldsymbol{e_3}$, $\vec{w}\bullet\vec{n}$ represents the component of $\vec{w}$ in the direction of $\vec{n}$, and dv is the volume bounded by the region D.

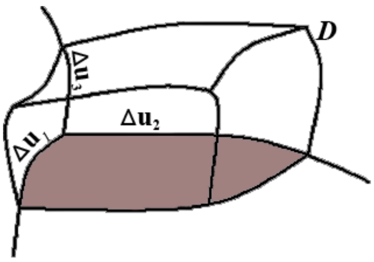

Fig. 1. An infinitesimal control volume bounded by the surface.

Considering the bottom shaded surface dS of the given control volume, Fig. 1, one can find that

$$\vec{n} = -\frac{\dfrac{\partial\vec{r}}{\partial u_1}\times\dfrac{\partial\vec{r}}{\partial u_2}}{\left|\dfrac{\partial\vec{r}}{\partial u_1}\times\dfrac{\partial\vec{r}}{\partial u_2}\right|}, \text{ and } dS = \left|\frac{\partial\vec{r}}{\partial u_1}\times\frac{\partial\vec{r}}{\partial u_2}\right| du_1 du_2 \tag{9}$$

Hence

$$\begin{aligned}\vec{w}\bullet\vec{n}\, dS &= -\vec{w}\bullet\frac{\dfrac{\partial\vec{r}}{\partial u_1}\times\dfrac{\partial\vec{r}}{\partial u_2}}{\left|\dfrac{\partial\vec{r}}{\partial u_1}\times\dfrac{\partial\vec{r}}{\partial u_2}\right|}\left|\frac{\partial\vec{r}}{\partial u_1}\times\frac{\partial\vec{r}}{\partial u_2}\right| du_1 du_2 \\ &= -(w_1\boldsymbol{e_1} + w_2\boldsymbol{e_2} + w_3\boldsymbol{e_3})\bullet\left(h_1\boldsymbol{e_1}\times h_2\boldsymbol{e_2}\right)du_1 du_2 \\ &= -w_3\left(\boldsymbol{e_3}\bullet\boldsymbol{e_1}\times\boldsymbol{e_2}\right)h_1 h_2\, du_1 du_2\end{aligned} \tag{10}$$

On the upper surface, using Taylor series, the outward flux is

$$w_3\left(\boldsymbol{e}_3\bullet\boldsymbol{e_1}\times\boldsymbol{e_2}\right)h_1h_2du_1du_2 + \frac{\partial}{\partial u_3}\left[w_3\left(\boldsymbol{e}_3\bullet\boldsymbol{e_1}\times\boldsymbol{e_2}\right)h_1h_2\right]du_1du_2du_3 + \ldots$$

The net flux through the upper and lower surfaces, then, is

$$\frac{\partial}{\partial u_3}\left[\left(\boldsymbol{e_3}\bullet\boldsymbol{e_1}\times e_2\right)w_3h_1h_2\right]du_1du_2du_3 + \cdots \tag{11}$$

Using the same argument, the net flux through the remaining two pairs of surfaces are:

$$\frac{\partial}{\partial u_2}\left[\left(\boldsymbol{e_3}\bullet\boldsymbol{e_1}\times e_2\right)w_2h_1h_3\right]du_1du_2du_3 + \cdots$$

$$\frac{\partial}{\partial u_1}\left[\left(\boldsymbol{e_3}\bullet\boldsymbol{e_1}\times e_2\right)w_1h_2h_3\right]du_1du_2du_3 + \cdots$$

$\vec{w}\bullet\vec{n}\, dS$ on the left hand side of equation (8) becomes

$$\left(\frac{\partial}{\partial u_3}\left[ew_3h_1h_2\right] + \frac{\partial}{\partial u_2}\left[ew_2h_1h_3\right] + \frac{\partial}{\partial u_1}\left[ew_1h_2h_3\right]\right) \times du_1du_2du_3 + \cdots \tag{12}$$

where $e = \boldsymbol{e_3}\bullet\boldsymbol{e_1}\times e_2$ is an *orthognality indicator* of the system, $(0 < e \le 1)$.

The volume element, dv, is given by

$$dv = \left(h_3\boldsymbol{e_3}\, du_3\right)\bullet\left(h_1\boldsymbol{e_1}\, du_1 \times h_2\boldsymbol{e_2}\, du_2\right) \tag{13}$$

The right hand side of equation (8), then, is

$$\begin{aligned}\vec{\nabla}\bullet\vec{w}\, dv &= \vec{\nabla}\bullet\vec{w}\left(h_3\boldsymbol{e_3}\, du_3\right)\bullet\left(h_1\boldsymbol{e_1}\, du_1\times h_2\boldsymbol{e_2}\, du_2\right) \\ &= \vec{\nabla}\bullet\vec{w}\left(\boldsymbol{e_3}\bullet\boldsymbol{e_1}\times e_2\right)\left(h_1h_2h_3du_1du_2du_3\right) \\ &= \vec{\nabla}\bullet\vec{w}\, e\left(h_1h_2h_3\, du_1du_2du_3\right)\end{aligned} \tag{14}$$

In the limit, by equating (12) and (14), an expression for the divergence of a vector field $\vec{w}$ in a general coordinates system, whether orthogonal or not, is given by

$$\vec{\nabla}\bullet\vec{w} = \frac{1}{e\,h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(e\ w_1h_2h_3) + \frac{\partial}{\partial u_2}(e\ w_2h_1h_3) + \frac{\partial}{\partial u_3}(e\ w_3h_1h_2)\right] \tag{15}$$

It is easy to see that when $e = 1$ (orthogonal system), equation (15) reduces to (5).

III. Curl for General Coordinates Systems

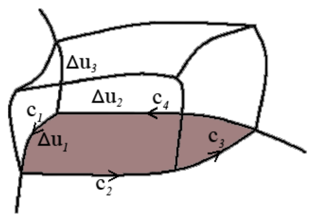

Fig. 2. An infinitesimal surface enclosed by the closed path c.

Stokes' Theorem relates the surface integral of the curl of a vector field over a surface to the line integral of the vector field over the surface boundary, or

$$\iint_S \vec{\nabla}\times\vec{w}\bullet\vec{n}\ dS = \oint_c \vec{w}\bullet d\vec{r} \tag{16}$$

Considering the bottom shaded surface of the given control volume as shown in Fig. 2,

This surface is bounded by the curve c which is traced counterclockwise. This curve is comprised of the four segments c_1, c_2, c_3, and c_4. We evaluate the right hand side and the left hand side of (16) over this surface and two other surfaces not opposite to each other. On the shown shaded surface, one can write

$$\begin{aligned}
&\vec{\nabla}\times\vec{w}\bullet\vec{n}\,dS = \\
&\left((\vec{\nabla}\times\vec{w})_1\boldsymbol{e_1}+(\vec{\nabla}\times\vec{w})_2\boldsymbol{e_2}+(\vec{\nabla}\times\vec{w})_3\boldsymbol{e_3}\right)\bullet \\
&\qquad\left(\frac{h_1\boldsymbol{e_1}\times h_2\boldsymbol{e_2}}{\left|h_1\boldsymbol{e_1}\times h_2\boldsymbol{e_2}\right|}\right)\left|h_1\boldsymbol{e_1}\times h_2\boldsymbol{e_2}\right|du_1du_2 \qquad (17)\\
&= \left((\vec{\nabla}\times\vec{w})_3\boldsymbol{e_3}\right)\bullet\left(\boldsymbol{e_1}\times\boldsymbol{e_2}\right)h_1h_2\ du_1\ du_2 \\
&= (\vec{\nabla}\times\vec{w})_3\ e\ h_1h_2du_1du_2
\end{aligned}$$

where $(\vec{\nabla}\times\vec{w})_i$ is the component of $\vec{\nabla}\times\vec{w}$ in the direction of $\boldsymbol{e_i}$.

The term $\vec{w}\bullet dr$ in equation (16), over the four segments comprising the curve c, is

$$\begin{aligned}
\vec{w}\bullet d\vec{r} &= \vec{w}\bullet d\vec{r}\big|_{c_1}+\vec{w}\bullet d\vec{r}\big|_{c_2}+\vec{w}\bullet d\vec{r}\big|_{c_3}+\vec{w}\bullet d\vec{r}\big|_{c_4} \\
&= \vec{w}\bullet d\vec{r}\big|_{c_1}+\vec{w}\bullet d\vec{r}\big|_{c_3}+\vec{w}\bullet d\vec{r}\big|_{c_4}+\vec{w}\bullet d\vec{r}\big|_{c_2} \\
&= \left(\vec{w}\bullet h_1\boldsymbol{e_1}du_1\right)+ \\
&\left(-\vec{w}\bullet h_1\,\boldsymbol{e_1}\,du_1+\frac{\partial}{\partial u_2}\left[-\vec{w}\bullet h_1\boldsymbol{e_1}du_1\right]du_2\right)+\cdots+ \\
&\left(-\vec{w}\bullet h_2\boldsymbol{e_2}\,du_2\right)+ \\
&\left(\vec{w}\bullet h_2\boldsymbol{e_2}\,du_2+\frac{\partial}{\partial u_1}\left[\vec{w}\bullet h_2\boldsymbol{e_2}\,du_2\right]du_1\right)+\cdots
\end{aligned}$$

Or

$$\vec{w}\bullet dr = \left(\frac{\partial}{\partial u_1}\left[\vec{w}\bullet h_2\boldsymbol{e_2}\right]-\frac{\partial}{\partial u_2}\left[\vec{w}\bullet h_1\boldsymbol{e_1}\right]\right)du_1du_2+\cdots \tag{18}$$

In the limit, after equating (17) and (18),

$$(\vec{\nabla}\times\vec{w})_3 = \frac{1}{e\ h_1h_2}\left(\frac{\partial}{\partial u_1}[h_2\vec{w}\bullet\boldsymbol{e_2}]-\frac{\partial}{\partial u_2}[h_1\vec{w}\bullet\boldsymbol{e_1}]\right) \tag{19}$$

Similar results can be obtained for the other two surfaces.

$$(\vec{\nabla}\times\vec{w})_2 = \frac{1}{e\ h_3h_1}\left(\frac{\partial}{\partial u_3}[h_1\vec{w}\bullet\boldsymbol{e}_1]-\frac{\partial}{\partial u_1}[h_3\vec{w}\bullet\boldsymbol{e}_3]\right) \tag{20}$$

$$(\vec{\nabla}\times\vec{w})_1 = \frac{1}{e\ h_3h_2}\left(\frac{\partial}{\partial u_2}[h_3\vec{w}\bullet\boldsymbol{e_3}]-\frac{\partial}{\partial u_3}[h_2\vec{w}\bullet\boldsymbol{e_2}]\right) \tag{21}$$

Equations (19), (20) and (21) can be written in the concise form

$$\vec{\nabla}\times\vec{w} = \frac{1}{e\,h_1h_2h_3}\begin{vmatrix} h_1\boldsymbol{e_1} & h_2\boldsymbol{e_2} & h_3\boldsymbol{e_3} \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1\vec{w}\bullet\boldsymbol{e_1} & h_2\vec{w}\bullet\boldsymbol{e_2} & h_3\vec{w}\bullet\boldsymbol{e_3}\end{vmatrix} \tag{22}$$

Obviously, the difference between orthogonal and nonorthogonal coordinates systems lies in the introduction of the scalar triple product $e=\boldsymbol{e_3}\bullet\boldsymbol{e_1}\times e_2$ into the expressions of $\vec{\nabla}\times\vec{F}$, and $\vec{\nabla}\bullet\vec{F}$. This product returns a value of unity if the coordinates system is orthogonal.

IV. Example

Consider the simple right-handed coordinates system (u,v,z) which is related to the Cartesian system by the relations

$$\begin{aligned} x &= u-v \\ y &= u \\ z &= z \end{aligned} \tag{23}$$

This system is composed of the planes:

$y=u$ which are planes parallel to the $x-z$ plane, $y=x+v$ which are planes that are extrusions of the straight lines $y=x+u$ into the z -direction, and planes that are parallel to the $x-y$ plane.

The level curves of the coordinates system on the $x-y$ plane are shown in Fig. 3.

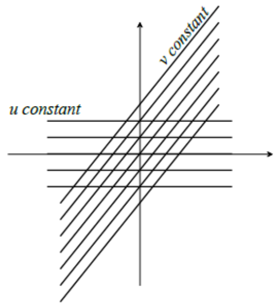

Fig. 3. Example of a non-orthogonal coordinates system.

Obviously, the coordinates system is not orthogonal. Simple calculations reveal that

$$\begin{aligned} h_1 &= h_u = \left|\frac{\partial\vec{r}}{\partial u}\right| = \sqrt{2} \\ h_2 &= h_v = \left|\frac{\partial\vec{r}}{\partial v}\right| = 1 \\ h_3 &= h_z = \left|\frac{\partial\vec{r}}{\partial z}\right| = 1 \end{aligned} \tag{24}$$

The corresponding unit vectors are

$$\boldsymbol{e_1} = \frac{\partial \vec{r}}{\partial u} \Big/ \left|\frac{\partial \vec{r}}{\partial u}\right| = (i+j)/\sqrt{2}$$
$$\boldsymbol{e_2} = \frac{\partial \vec{r}}{\partial v} \Big/ \left|\frac{\partial \vec{r}}{\partial v}\right| = -i \qquad (25)$$
$$\boldsymbol{e_3} = \frac{\partial \vec{r}}{\partial z} \Big/ \left|\frac{\partial \vec{r}}{\partial z}\right| = k$$

Now, the orthogonality indicator is

$$e = \boldsymbol{e_3} \bullet \boldsymbol{e_1} \times e_2 = \begin{vmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -1 & 0 & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \qquad (26)$$

The value of the orthogonality indicator being different from unity indicates that the system is indeed skew.

One can now find the corresponding operator (15) as

$$\vec{\nabla} \bullet \vec{F} = \frac{1}{\sqrt{2}} \frac{\partial F_1}{\partial u_1} + \frac{\partial F_2}{\partial u_2} + \frac{\partial F_3}{\partial u_3} \qquad (27)$$

and the three components of the curl in (22) to be

$$(\vec{\nabla} \times \vec{w})_1 = \sqrt{2}\left(\frac{\partial w_3}{\partial v} - \frac{\partial (w_2 - w_1/\sqrt{2})}{\partial z}\right) \qquad (28)$$

$$(\vec{\nabla} \times \vec{w})_2 = \sqrt{2}\,\frac{\partial (w_1 - w_2/\sqrt{2})}{\partial z} - \frac{\partial w_3}{\partial u} \qquad (29)$$

$$(\vec{\nabla} \times \vec{w})_3 = \frac{\partial (-w_1/\sqrt{2} + w_2)}{\partial u} - \sqrt{2}\,\frac{\partial (w_1 - w_2/\sqrt{2})}{\partial v} \qquad (30)$$

V. Conclusion

Two formulas for the divergence and curl operators in any coordinates systems whether orthogonal or not have been obtained. These formulas generalize the well known and widely used relations for orthogonal coordinates systems. The difference between orthogonal and nonorthogonal coordinates systems lies in the introduction of the orthogonality indicator $e = \boldsymbol{e_3} \bullet \boldsymbol{e_1} \times e_2$ which returns a value of unity if the coordinates system is orthogonal.

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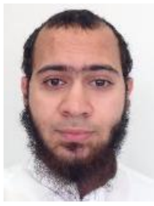
Rajai S. Alassar is a professor in the Department of Mathematics and Statistics at King Fahd University of Petroleum & Minerals (KFUPM). He was born in Gaza in 1965. He holds a PhD in hydrodynamics.

He has published many papers in his field of interest and has authored 2 textbooks in Arabic on the subject of Math Olympiad. He has also translated the famous book of Zill on differential equations to Arabic.



Mohammed Abushoshah is a lecturer in the Department of Mathematics and Statistics at King Fahd University of Petroleum & Minerals (KFUPM). He was born in Jeddah in 1987. He holds an MSc in applied mathematics.

He has published several papers in applied mathematics with concentration on fluid mechanics and thermal sciences.