Divergence and Curl Operators in Skew Coordinates

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Abstract—The divergence and curl operators appear in numerous differential equations governing engineering and physics problems. These operators, whose forms are well known in general orthogonal coordinates systems, assume different casts in different systems. In certain instances, one needs to custom-make a coordinates system that may turn out to be skew (i.e. not orthogonal). Of course, the known formulas for the divergence and curl operators in orthogonal coordinates are not useful in such cases, and one needs to derive their counterparts in skew systems. In this note, we derive two formulas for the divergence and curl operators in a general coordinates system, whether orthogonal or not. These formulas generalize the well known and widely used relations for orthogonal coordinates systems. In the process, we define an orthogonality indicator whose value ranges between zero and unity.

Index Terms—Coordinates systems, curl, divergence, Laplace, skew systems.

I. INTRODUCTION

The curl and divergence operators play significant roles in physical relations. They arise in fluid mechanics, elasticity theory and are fundamental in the theory of electromagnetism, [1], [2].

The physical significance of the Curl of a vector field \( \mathbf{F} \), denoted by \( \nabla \times \mathbf{F} \), is that it measures the amount of rotation or angular momentum of the contents of a given region of space. If the value of the curl is zero then the field is said to be irrotational. The curl is defined in an arbitrary orthogonal curvilinear coordinates \((u_1, u_2, u_3)\) as

\[
\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix}
 h_1 e_1 & h_2 e_2 & h_3 e_3 \\
 \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\
 F_1 & F_2 & F_3
\end{vmatrix}
\]

where \( e_i \) is a unit vector in the direction of \( u_i \), and \( \mathbf{F} = F_1 e_1 + F_2 e_2 + F_3 e_3 \). The length of the tangent vector in the direction of \( u_i \) is known as scale factor, \( h_i \), and is defined by

\[
h_i = \left| \frac{\partial \mathbf{r}}{\partial u_i} \right|
\]

where \( \mathbf{r} \) is the position vector in any three dimensional space, i.e. \( \mathbf{r} = \mathbf{r}(u_1, u_2, u_3) \).

Note that \( \frac{\partial \mathbf{r}}{\partial u_i} \) is a tangent vector to the \( u_i \) curve where the other two coordinates variables remain constant. A unit tangent vector in this direction, therefore, is

\[
e_i = \frac{\frac{\partial \mathbf{r}}{\partial u_i}}{h_i} = \frac{\partial \mathbf{r}}{h_i} \quad \text{or} \quad \frac{\partial \mathbf{r}}{\partial u_i} = h_i e_i
\]

The Divergence of a vector field over a control volume \( V \) bounded by the surface \( S \), denoted by \( \nabla \cdot \mathbf{F} \), is defined by

\[
\nabla \cdot \mathbf{F} = \lim_{V \to 0} \frac{\oint_{\partial V} \mathbf{F} \cdot \mathbf{n} \, dS}{V}
\]

where \( \mathbf{n} \) is an outward unit normal vector to the surface \( S \). The divergence actually measures the net outflow of a vector field from an infinitesimal volume around a given point (or how much a vector field "converges to" or "diverges from" a given point). The general expression of the divergence for arbitrary orthogonal curvilinear coordinates is given by

\[
\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} (F_2 h_3 h_1) + \frac{\partial}{\partial u_2} (F_3 h_2 h_1) + \frac{\partial}{\partial u_3} (F_1 h_2 h_3) \right]
\]

The Laplacian operator of a scalar \( \psi = \psi(u_1, u_2, u_3) \), denoted by \( \nabla^2 \psi \) is defined as the divergence of the gradient of \( \psi \); that is

\[
\nabla^2 \psi = \nabla \cdot (\nabla \psi) = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} \left( \frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial u_3} \right) \right]
\]

where

\[
\nabla \psi = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} e_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial u_2} e_2 + \frac{1}{h_3} \frac{\partial \psi}{\partial u_3} e_3
\]

Equations (1) and (5) are valid for orthogonal systems.
only (i.e. $e_1$, $e_2$, and $e_3$ are orthogonal), [3]-[5].

II. DIVERGENCE FOR GENERAL COORDINATES SYSTEMS

The Divergence Theorem relates the flow, or the flux, of a vector field through a surface to the behavior of the vector field inside the surface. More precisely, it states that the outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface, i.e.

$$\iiint_{S} \mathbf{w} \cdot d\mathbf{S} = \iiint_{D} \nabla \cdot \mathbf{w} \, dv$$  \hspace{1cm} (8)

where $D$ is a closed bounded region with piecewise smooth boundary $S$, $\mathbf{n}$ is an outer unit vector normal to the surface $S$, $w = w_1 e_1 + w_2 e_2 + w_3 e_3$, $\mathbf{w} \cdot \mathbf{n}$ represents the component of $\mathbf{w}$ in the direction of $\mathbf{n}$, and $dv$ is the volume bounded by the region $D$.

![Fig. 1. An infinitesimal control volume bounded by the surface.](image1)

Considering the bottom shaded surface $dS$ of the given control volume, Fig. 1, one can find that

$$\mathbf{n} = \left[\frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}\right], \text{ and } dS = \left|\frac{\partial \mathbf{r}}{\partial u_1} \times \frac{\partial \mathbf{r}}{\partial u_2}\right| \, du_1 du_2$$  \hspace{1cm} (9)

Hence

$$\mathbf{w} \cdot d\mathbf{S} = -\mathbf{w} \cdot \mathbf{n} \, dS = -(w_1 e_1 + w_2 e_2 + w_3 e_3) \cdot (h_1 e_1 + h_2 e_2) \, du_1 du_2$$

$$= -(w_1 e_1 + w_2 e_2 + w_3 e_3) \cdot (h_1 e_1 + h_2 e_2) \, du_1 du_2$$

$$= -w_1 (e_1 \cdot e_1) h_1 h_2 \, du_1 du_2$$

On the upper surface, using Taylor series, the outward flux is

$$w_3 (e_3 \cdot e_1) h_3 h_2 \, du_1 du_2 + \frac{\partial}{\partial u_3} \left[ w_3 (e_3 \cdot e_1) h_3 h_2 \right] du_1 du_2 du_3 + \cdots$$

The net flux through the upper and lower surfaces, then, is

$$\frac{\partial}{\partial u_3} \left[ w_3 (e_3 \cdot e_1) h_3 h_2 \right] du_1 du_2 du_3 + \cdots$$  \hspace{1cm} (11)

Using the same argument, the net flux through the remaining two pairs of surfaces are:

$$\frac{\partial}{\partial u_2} \left[ w_2 (e_2 \cdot e_1) h_2 h_3 \right] du_1 du_2 du_3 + \cdots$$

$$\frac{\partial}{\partial u_1} \left[ w_1 (e_1 \cdot e_1) h_1 h_2 \right] du_1 du_2 du_3 + \cdots$$

$\mathbf{w} \cdot d\mathbf{S}$ on the left hand side of equation (8) becomes

$$\left( \frac{\partial}{\partial u_3} [e w_3 h_2 h_3] + \frac{\partial}{\partial u_2} [e w_2 h_3] + \frac{\partial}{\partial u_1} [e w_1 h_2] \right) \times du_1 du_2 du_3 + \cdots$$  \hspace{1cm} (12)

where $e = e_3 \cdot e_1 \times e_2$ is an orthognality indicator of the system, $0 < e \leq 1$.

The volume element, $dv$, is given by

$$dv = (h_1 e_3 du_3) (h_1 e_1 du_1 \times h_2 e_2 du_2)$$  \hspace{1cm} (13)

The right hand side of equation (8), then, is

$$\nabla \cdot \mathbf{w} \cdot \mathbf{w} \, dv = \nabla \cdot \mathbf{w} \cdot (h_1 e_3 du_3) (h_1 e_1 du_1 \times h_2 e_2 du_2)$$

$$= \nabla \cdot \mathbf{w} (e_3 \cdot e_1 \times e_2) (h_1 h_3 h_2 du_1 du_2 du_3)$$

$$= \nabla \cdot \mathbf{w} e (h_1 h_2 h_3 du_1 du_2 du_3)$$

In the limit, by equating (12) and (14), an expression for the divergence of a vector field $\mathbf{w}$ in a general coordinates system, whether orthogonal or not, is given by

$$\nabla \cdot \mathbf{w} = \frac{1}{e h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} (e w_1 h_2 h_3) + \frac{\partial}{\partial u_2} (e w_2 h_3) + \frac{\partial}{\partial u_3} (e w_3 h_2 h_3) \right]$$  \hspace{1cm} (15)

It is easy to see that when $e = 1$ (orthogonal system), equation (15) reduces to (5).

III. CURL FOR GENERAL COORDINATES SYSTEMS

![Fig. 2. An infinitesimal surface enclosed by the closed path c.](image2)
Stokes’ Theorem relates the surface integral of the curl of a vector field over a surface to the line integral of the vector field over the surface boundary, or
\[
\int_S \nabla \times \mathbf{w} \cdot dS = \oint_c \mathbf{w} \cdot d\mathbf{r} \tag{16}
\]

Considering the bottom shaded surface of the given control volume as shown in Fig. 2, this surface is bounded by the curve \( c \) which is traced counterclockwise. This curve is comprised of the four segments \( c_1, c_2, c_3, \) and \( c_4 \). We evaluate the right hand side and the left hand side of (16) over this surface and two other surfaces not opposite to each other. On the shown shaded surface, one can write

\[
(\nabla \times \mathbf{w})_3 = \frac{1}{e} \left( \frac{\partial}{\partial u_1} \left[ h_2 \mathbf{w} \cdot \mathbf{e}_2 \right] - \frac{\partial}{\partial u_2} \left[ h_1 \mathbf{w} \cdot \mathbf{e}_1 \right] \right) \tag{19}
\]

Similar results can be obtained for the other two surfaces.

\[
(\nabla \times \mathbf{w})_2 = \frac{1}{e} \left( \frac{\partial}{\partial u_2} \left[ h_3 \mathbf{w} \cdot \mathbf{e}_3 \right] - \frac{\partial}{\partial u_3} \left[ h_2 \mathbf{w} \cdot \mathbf{e}_2 \right] \right) \tag{20}
\]

Equations (19), (20) and (21) can be written in the concise form

\[
\nabla \times \mathbf{w} = \frac{1}{e h_1 h_2 h_3} \begin{vmatrix}
  h_1 e_1 & h_2 e_2 & h_3 e_3 \\
  \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\
  h_1 \mathbf{w} \cdot \mathbf{e}_1 & h_2 \mathbf{w} \cdot \mathbf{e}_2 & h_3 \mathbf{w} \cdot \mathbf{e}_3 
\end{vmatrix} \tag{22}
\]

This system is composed of the planes:

\( y = u \) which are planes parallel to the \( x-z \) plane, and
\( y = x+u \) which are planes that are extrusions of the straight lines \( y = x \) into the \( z \)-direction, and planes that are parallel to the \( x-y \) plane.

The level curves of the coordinates system on the \( x-y \) plane are shown in Fig. 3.

\[
\begin{align*}
\text{V. Example} & \\
\text{Consider the simple right-handed coordinates system } & (u, v, z) \text{ which is related to the Cartesian system by the relations} \\
x &= u - v \\
y &= u \\
z &= z \\
\end{align*}
\]

\[
\text{Or} \quad \mathbf{w} \cdot d\mathbf{r} = \left( \frac{\partial}{\partial u_1} \left[ \mathbf{w} \cdot \mathbf{e}_1 \right] - \frac{\partial}{\partial u_2} \left[ \mathbf{w} \cdot \mathbf{e}_2 \right] \right) du_1 du_2 + \cdots 
\]

\[
\text{In the limit, after equating (17) and (18),} \\
(\nabla \times \mathbf{w})_3 = \frac{1}{e h_1 h_2 h_3} \left( \frac{\partial}{\partial u_1} \left[ h_2 \mathbf{w} \cdot \mathbf{e}_2 \right] - \frac{\partial}{\partial u_2} \left[ h_1 \mathbf{w} \cdot \mathbf{e}_1 \right] \right) \tag{19}
\]

\[
\text{Similar results can be obtained for the other two surfaces.} \\
(\nabla \times \mathbf{w})_2 = \frac{1}{e h_1 h_2} \left( \frac{\partial}{\partial u_2} \left[ h_3 \mathbf{w} \cdot \mathbf{e}_3 \right] - \frac{\partial}{\partial u_3} \left[ h_2 \mathbf{w} \cdot \mathbf{e}_2 \right] \right) \tag{20}
\]

\[
(\nabla \times \mathbf{w})_1 = \frac{1}{e h_1 h_2} \left( \frac{\partial}{\partial u_2} \left[ h_1 \mathbf{w} \cdot \mathbf{e}_1 \right] - \frac{\partial}{\partial u_1} \left[ h_3 \mathbf{w} \cdot \mathbf{e}_3 \right] \right) \tag{21}
\]
The corresponding unit vectors are

\[ \mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial u} = \frac{i + j}{\sqrt{2}} \]
\[ \mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial v} = -i \]
\[ \mathbf{e}_3 = \frac{\partial \mathbf{r}}{\partial z} = k \]

(25)

Now, the orthogonality indicator is

\[ e = \mathbf{e}_3 \cdot \mathbf{e}_1 \times \mathbf{e}_2 = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{vmatrix} = \frac{1}{\sqrt{2}} \]

(26)

The value of the orthogonality indicator being different from unity indicates that the system is indeed skew.

One can now find the corresponding operator (15) as

\[ \mathbf{\nabla} \times \mathbf{F} = \frac{1}{\sqrt{2}} \frac{\partial F_3}{\partial u} + \frac{\partial F_2}{\partial u_2} + \frac{\partial F_1}{\partial u_3} \]

(27)

and the three components of the curl in (22) to be

\[ (\mathbf{\nabla} \times \mathbf{w})_1 = \sqrt{2} \left( \frac{\partial w_3}{\partial v} - \frac{\partial (w_2 - w_1 / \sqrt{2})}{\partial z} \right) \]

(28)

\[ (\mathbf{\nabla} \times \mathbf{w})_2 = \sqrt{2} \left( \frac{\partial (w_1 - w_2 / \sqrt{2})}{\partial z} - \frac{\partial w_3}{\partial u} \right) \]

(29)

\[ (\mathbf{\nabla} \times \mathbf{w})_3 = \frac{\partial (w_3 - \sqrt{2} w_1)}{\partial u} - \sqrt{2} \left( \frac{\partial (w_1 - w_2 / \sqrt{2})}{\partial v} \right) \]

(30)

V. CONCLUSION

Two formulas for the divergence and curl operators in any coordinates systems whether orthogonal or not have been obtained. These formulas generalize the well known and widely used relations for orthogonal coordinates systems. The difference between orthogonal and nonorthogonal coordinates systems lies in the introduction of the orthogonality indicator \( e = \mathbf{e}_3 \cdot \mathbf{e}_1 \times \mathbf{e}_2 \) which returns a value of unity if the coordinates system is orthogonal.

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