On Calculating Method of the Kelly Criterion for Financial Investment in Single Risky Asset with Various Distributions of Returns

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***Abstract*—In this paper, the expectation of the reciprocal of first-degree polynomials of non-negative valued random variables is calculated. This is motivated to compute the Kelly criterion, which is the optimal solution of the maximization of the expected logarithm of the investment return. As soon as the expectation of the reciprocal of first-degree polynomials of asset returns is calculated, which is our main interest, the Kelly criterion can be obtained by using the ordinary optimization technique or applying the appropriate algorithm.**

***Index Terms*—Expectation, probability, investment strategy, Kelly criterion.**

I. Introduction

This paper aims to exhibit some concrete calculation of $E[1/(c+X)]$ where E denotes the expectation of random variables, X is a nonnegative valued random variable, and $c>0$. This is motivated by the maximization problem

$$\max_{0\le b\le 1} E[\log(1-b+b\,X)\,]. \quad (1)$$

This problem can be solved by searching b^* at which the derivative of $E[\log(1-b+b\,X)]$ with respect to b equals zero:

$$\frac{d}{db}E[\log(1-b^*+\ b^*X)] = E\left[\frac{-1+X}{1-b^*+b^*\,X}\right]=0. \quad (2)$$

Since

$$E\left[\frac{-1+X}{1-b^*+b^*X}\right]=\frac{1}{b^*}-\frac{1}{(b^*)^2}E\left[\frac{1}{\frac{1-b^*}{b^*}+X}\right],$$

calculating $E[1/(c+X)]$, $c>0$ is critical to solve the problem (1). Another solution to the problem (1) is utilization of the *Cover's algorithm* developed by [1]. According to [1], the optimal solution b^* is obtained by $b^*=\lim_{n\to\infty} b^{(n)}$ where $b^{(n)}, n=0,1,2,\ldots$ are defined recurrently by

$$b^{(0)}=\frac{1}{2},$$

$$b^{(n+1)}=E\left[\frac{b^{(n)}X}{1-b^{(n)}+b^{(n)}X}\right], \qquad n=1,2,\ldots. \quad (3)$$

It is clear that calculation of $E[1/(c+X)], c>0$ is also essential even in this case.

This problem often arises in financial economics. The problem (1) can be regarded as the maximization of the expected logarithmic return of the investment strategy that allocates the proportion b of one's wealth to the risky asset with return X and saves the remaining $1-b$ of the wealth. The solution b^* of this problem is called the *Kelly criterion* [2]-[5]. The explicit derivation of the Kelly criterion is thought of as a hard problem because of the nonlinearity of the problem (1) and the complexity of the probability distribution of X unless X is a Bernoulli random variable. Then, in order to calculate the Kelly criterion, some approximate solution is often taken. See, e.g., [6], [7]. These approximations are done by expanding the logarithm in (1) to the second-order as the Taylor series. Other investors may use a simplified Kelly criterion called the *fortune's formula*, which is an allocation rule defied by $\hat{b}=edge/odds$ and corresponds to the Kelly criterion when the return X is Bernoulli distributed (see, e.g., [8]). However, [9] has shown that the use of the simplified Kelly criterion $\hat{b}$ instead of the exact Kelly criterion b^* may lead to ruin. Then, it turns out that it is worth providing a more accurate calculation method of b^*. The present paper asserts that as soon as $E[1/(c+X)]$ is calculated, more accurate approximate solutions become available by solving (2) with respect to b^* numerically or by utilizing the Cover's algorithm (3).

In this paper, we will calculate $E[1/(c+X)]$ when X is (i) the square of a Cauchy random variable, (ii) the absolute value of a Cauchy random variable, (iii) Pareto Type 2 distributed with integer shape parameters, (iv) log-normal distributed, (v) Rayleigh distributed and (vi) Erlang distributed (or gamma distributed with integer shape parameters). The calculations in this paper can be divided into three different methods.

a) The first method is done by simply calculating the *integral expression* of $E[1/(c+X)]$. The cases from (i) to (iv) above are done by this method.
b) The second method uses the *ordinary differential*

Manuscript received October 24, 2020; revised December 23, 2020. The author is supported by a fellowship of Japan Society for the Promotion of Science (JSPS). This research is also supported by JSPS KAKENHI Grant Number 18J00193.

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DOI: 10.7763/IJMO.2021.V11.775

equation (ODE). We define a function

$$g(z) = E\left[\frac{1}{z+X}\right].$$

Then, we derive the ODE that $g(z)$ satisfies, solve the ODE, and evaluate $g(c)$. The Rayleigh case (v) is treated by this method.

c) The third method utilizes *mixed distributions and the Laplace transform*. That is, if we introduce a random variable Z such that $Z|X \sim \mathrm{Exp}(X)$, then we realize that
$E[e^{-cZ}] = E\left[E[e^{-cZ}|X]\right] = 1 - c\,E[1/(c+X)].$

Then, the problem becomes the calculation of the Laplace transform $E[e^{-cZ}]$ for the unconditioned probability distribution of Z. The Erlang case (vi) is dealt with by this method.

The rest of the paper is organized as follows. The calculation results of $E[1/(c+X)]$ are summarized in section II. In section III, the proofs are given separately for each method. Section IV concludes the paper.

A. A List of Special Functions

In calculations below, we use the following special functions.

- The exponential integral $E_n(z)$:

$$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n}dt, \qquad n = 0,1,2,\dots, \mathrm{Re}(z) > 0. \quad (4)$$

See eq.5.1.4, p.228 in [10].

- The exponential integral $\mathrm{Ei}(x)$:

$$\mathrm{Ei}\,(x) = \int_{-\infty}^{x} \frac{e^t}{t}dt\,, \qquad x > 0, \quad (5)$$

See eq.5.1.2, p.228 in [10].

- The error function $erf\ (z)$:

$$\mathrm{erf(z)} = \frac{2}{\sqrt{\pi}}\int_0^{\mathrm{z}} e^{-t^2}\,\mathrm{dt}. \quad (6)$$

See eq.7.1.1, p.297 in [10].

II. Results

Proposition 1. Suppose that X is a Cauchy random variable whose probability density function (PDF) is given by

$$f_X(x) = \frac{1}{\pi}\cdot\frac{\gamma}{\gamma^2+x^2}, \qquad -\infty < x < \infty, \gamma > 0.$$

Then, for $c > 0$,

$$E\left[\frac{1}{c+X^2}\right] = \frac{1}{\sqrt{c}\left(\gamma+\sqrt{c}\right)}. \quad (7)$$

Proposition 2. *Suppose that X is a Cauchy* distributed *random variable whose PDF is given by* $f_X(x) = \frac{1}{\pi}\cdot\frac{\gamma}{\gamma^2+x^2}$, $-\infty < x < \infty,\ \gamma > 0$. *Then, for $c > 0$,*

$$E\left[\frac{1}{c+|X|}\right] = \frac{c}{\gamma^2+c^2} + \frac{2\gamma}{\pi(\gamma^2+c^2)}(\log\gamma - \log c).$$

Proposition 3. *Suppose that X is a Pareto Type 2 distributed random variable with integer shape parameters whose PDF is given by*

$$f_X(x) = \frac{n\lambda^n}{(x+\lambda)^{n+1}}, \qquad x > 0, \lambda > 0, n = 1,2,\dots.$$

Then, for c > 0,

$$E\left[\frac{1}{c+X}\right] = -\frac{n}{\lambda}\left(\frac{\lambda}{\lambda-c}\right)^{n+1}\left(\log c - \log\lambda + \sum_{k=1}^{n}\frac{1}{k}\left(\frac{\lambda-c}{\lambda}\right)^k\right).$$

Proposition 4. Suppose that X is a log-normal distributed random variable such that $log\,X \sim N(0,\sigma^2)$ where $\sigma > 0$. Then, for c > 0,

$$E\left[\frac{1}{c+X}\right] = \frac{1}{c}\left\{\begin{array}{c}\bar{\Phi}\left(\frac{\log c}{\sigma}\right) \\ -\sum_{k=0}^{\infty}(-1)^k\exp\left(\frac{\sigma^2k^2}{2}+k\log c\right)\bar{\Phi}\left(\frac{\sigma^2k+\log c}{\sigma}\right) \\ +\sum_{k=0}^{\infty}(-1)^k\exp\left(\frac{\sigma^2k^2}{2}-k\log c\right)\bar{\Phi}\left(\frac{\sigma^2k-\log c}{\sigma}\right)\end{array}\right\},$$

where $\bar{\Phi}$ indicates the tail distribution of the standard normal distribution, i.e., $\bar{\Phi} = \mathrm{P(Z > z)},\ \mathrm{z} \in \mathbb{R}$ when $\mathrm{Z} \sim \mathrm{N(0,1)}$.

Proposition 5. *Suppose that X is a Rayleigh distributed random variable whose PDF is given by*

$$f_X(x) = \frac{x}{\theta^2}\exp\left(-\frac{x^2}{2\theta^2}\right), \qquad x > 0, \qquad \theta > 0.$$

Then, for $c > 0$,

$$E\left[\frac{1}{c+X}\right] = \sqrt{\frac{\pi}{2\theta^2}} + \frac{c}{2\theta^2}\exp\left(-\frac{c^2}{2\theta^2}\right)\cdot\left[\mathrm{Ei}\left(\frac{c^2}{2\theta^2}\right) - \pi\mathrm{i}\cdot\mathrm{erf}\left(-\frac{c}{\sqrt{2\theta^2}}\mathrm{i}\right)\right], \quad (8)$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, $\mathrm{Ei(x)}$ is the exponential integral defined by (5) and $\mathrm{erf\,(z)}$ is the error function defined by (6).

Proposition 6. Suppose that X is an Erlang distributed random variable whose PDF is given by

$$f_X(x) = \frac{1}{\Gamma(n)}\lambda^n x^{n-1}e^{-\lambda x}, \qquad x > 0, \lambda > 0, n = 1,2,\dots$$

Then, for $c > 0$,

$$E\left[\frac{1}{c+X}\right] = \lambda e^{c\lambda}E_n(c\lambda),$$

where $\mathrm{E_n(z)}$ is the exponential integral defined by (4).

III. Proofs

A. Using the Integral Expression

Proposition 1, 2, 3 and 4 can be obtained by simply calculating the integral form of $E[1/(c+X)]$.

Proof of Proposition *1*. For $c \neq \gamma^2$,

$$E\left[\frac{1}{c+X^2}\right] = \frac{\gamma}{\pi}\int_{-\infty}^{\infty}\frac{1}{c+x^2}\cdot\frac{1}{\gamma^2+x^2}dx$$
$$= \frac{2}{\pi}\cdot\frac{\gamma}{\gamma^2-c}\int_0^\infty\left(\frac{1}{c+x^2} - \frac{1}{\gamma^2+x^2}\right)dx$$

$$= \frac{2}{\pi} \cdot \frac{\gamma}{\gamma^2 - c}\left[\frac{1}{\sqrt{c}}\arctan\left(\frac{x}{\sqrt{c}}\right) - \frac{1}{\gamma}\arctan\left(\frac{x}{\gamma}\right)\right]_0^\infty$$
$$= \frac{2}{\pi} \cdot \frac{\gamma}{\gamma^2 - c} \cdot \frac{\pi}{2}\left(\frac{1}{\sqrt{c}} - \frac{1}{\gamma}\right)$$
$$= \frac{1}{(\gamma + \sqrt{c})\sqrt{c}}.$$

This is the right hand side of (7).

When $c = \gamma^2$,

$$E\left[\frac{1}{c + X^2}\right] = \frac{2\gamma}{\pi}\int_0^\infty \frac{1}{(\gamma^2 + x^2)^2}dx$$
$$= \frac{2\gamma}{\pi}\left[\frac{1}{2\gamma^2}\left(\frac{x}{\gamma^2 + x^2} + \frac{1}{\gamma}\arctan\left(\frac{x}{\gamma}\right)\right)\right]_0^\infty$$
$$= \frac{2\gamma}{\pi} \cdot \frac{\pi}{4\gamma^3}$$
$$= \frac{1}{2\gamma^2}.$$

This is the right hand side of (7) when $c = \gamma^2$. //

Proof of Proposition 2. By integration by parts,

$$E\left[\frac{1}{c + |X|}\right] = \frac{1}{\pi}\int_{-\infty}^\infty \frac{1}{c + |x|} \cdot \frac{\gamma}{\gamma^2 + x^2}dx$$
$$= \frac{2}{\pi}\int_0^\infty \frac{1}{c + x} \cdot \frac{\gamma}{\gamma^2 + x^2}dx$$
$$= \frac{2}{\pi} \cdot \frac{\gamma}{\gamma^2 + c^2}\int_0^\infty \left(\frac{1}{c + x} - \frac{x - c}{\gamma^2 + x^2}\right)dx$$
$$= \frac{2}{\pi} \cdot \frac{\gamma}{\gamma^2 + c^2}\left[\log\left(\frac{c + x}{\sqrt{\gamma^2 + x^2}}\right) + \frac{c}{\gamma}\arctan\left(\frac{x}{\gamma}\right)\right]_0^\infty$$
$$= \frac{2}{\pi} \cdot \frac{\gamma}{\gamma^2 + c^2}\left(\frac{c\pi}{2\gamma} - \log\left(\frac{c}{\gamma}\right)\right)$$

$= \frac{c}{\gamma^2+c^2} + \frac{2}{\pi} \cdot \frac{\gamma}{\gamma^2+c^2}(\log\gamma - \log c)$, and we get the claim. //

Proof of Proposition 3. First of all, recall that

$$\int_0^\infty e^{-st}\frac{1}{(t + a)^n}dt = a^{1-n}e^{as}E_n(as)$$

for $a > 0,\ n = 0,1,2,\dots$ and

$$\int_0^\infty e^{-at}E_{n+1}(t)\,\mathrm{d}t = \frac{(-1)^{\mathrm{n}}}{a^{n+1}}\left(\log(1 + a) + \sum_{k=1}^n \frac{(-1)^k a^k}{k}\right)$$

for $a > -1$ (see eq.29.3.128, p.1029 and eq.5.1.34, p.230 of [10]) where $E_n(z)$ is the exponential integral defined by (4). Then,

$$\int_0^\infty \frac{1}{c + x} \cdot \frac{1}{(x + \lambda)^{n+1}}dx$$
$$= \int_0^\infty\int_0^\infty e^{-(x+c)t}\frac{1}{(x + \lambda)^{n+1}}dt\,dx$$
$$= \int_0^\infty e^{-ct}dt\int_0^\infty e^{-xt}\frac{1}{(x + \lambda)^{n+1}}dx$$
$$= \int_0^\infty e^{-ct}\lambda^{-n}e^{\lambda t}E_{n+1}(\lambda t)\,dt$$
$$= \lambda^{-n-1}\int_0^\infty e^{-\frac{c-\lambda}{\lambda}s}E_{n+1}(s)\,ds$$
$$= \lambda^{-n-1}(-1)^n\left(\frac{\lambda}{c - \lambda}\right)^{n+1}\left(\log\left(1 + \frac{c - \lambda}{\lambda}\right) + \sum_{k=1}^n \frac{(-1)^k\left(\frac{c - \lambda}{\lambda}\right)^k}{k}\right)$$
$$= -\lambda^{-n-1}\left(\frac{\lambda}{\lambda - c}\right)^{n+1}\left(\log c - \log\lambda + \sum_{k=1}^n \frac{1}{k}\left(\frac{\lambda - c}{\lambda}\right)^k\right)$$

where the change of variable has been done as $s = \lambda t$ in the fourth equation. Note that, since $c > 0$ and $\lambda > 0$, it is always satisfied that $(c - \lambda)/\lambda > -1$. Therefore,

$$E\left[\frac{1}{c + X}\right] = \int_0^\infty \frac{1}{c + x} \cdot \frac{n\lambda^n}{(x + \lambda)^{n+1}}dx$$
$$= -\frac{n}{\lambda}\left(\frac{\lambda}{\lambda - c}\right)^{n+1}\left(\log c - \log\lambda + \sum_{k=1}^n \frac{1}{k}\left(\frac{\lambda - c}{\lambda}\right)^k\right),$$

and the proof is completed. //

Proof of Proposition 4. Denote $Y = \log X$ so that $Y \sim \mathrm{N}(0, \sigma^2)$. Then,

$$\mathrm{E}\left[\frac{1}{c + X}\right] = E\left[\frac{1}{c + e^Y}\right] = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^\infty \frac{1}{c + e^z}e^{-\frac{z^2}{2\sigma^2}}dz$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}c}\int_{-\infty}^\infty \frac{1}{c + e^{z-\log c}}e^{-\frac{z^2}{2\sigma^2}}dx. \tag{9}$$

By changing the variable as $y = z - \log c$,

$$= \frac{1}{\sqrt{2\pi\sigma^2}c}\int_{-\infty}^\infty \frac{1}{1 + e^y}e^{-\frac{(y+\log c)^2}{2\sigma^2}}dy$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}c}\int_0^\infty \frac{1}{1 + e^y}e^{-\frac{(y+\log c)^2}{2\sigma^2}}dy$$
$$+ \frac{1}{\sqrt{2\pi\sigma^2}c}\int_0^\infty \frac{1}{1 + e^{-y}}e^{-\frac{(y-\log c)^2}{2\sigma^2}}dy. \tag{10}$$

Since $\frac{1}{1 + e^y} = 1 - \frac{1}{1 + e^{-y}}$,

$$= \frac{1}{\sqrt{2\pi\sigma^2}c}\int_0^\infty e^{-\frac{(y+\log c)^2}{2\sigma^2}}dy \tag{11}$$
$$- \frac{1}{\sqrt{2\pi\sigma^2}c}\int_0^\infty \frac{1}{1 + e^{-y}}e^{-\frac{(y+\log c)^2}{2\sigma^2}}dy \tag{12}$$
$$+ \frac{1}{\sqrt{2\pi\sigma^2}c}\int_0^\infty \frac{1}{1 + e^{-y}}e^{-\frac{(y-\log c)^2}{2\sigma^2}}dy. \tag{13}$$

The first term is

$$(11) = \frac{1}{c}\overline{\Phi}\left(\frac{\log c}{\sigma}\right).$$

The second term (12) is calculated as

$$= \frac{1}{\sqrt{2\pi\sigma^2}c}\int_0^\infty \sum_{k=0}^\infty (-1)^k e^{-ky}e^{-\frac{(y+\log c)^2}{2\sigma^2}}dy$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}c}\sum_{k=0}^\infty (-1)^k\int_0^\infty e^{-ky}e^{-\frac{(y+\log c)^2}{2\sigma^2}}dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}c}\sum_{k=0}^{\infty}(-1)^k e^{\frac{\sigma^2k^2}{2}+k\log c}\int_0^{\infty} e^{-\frac{(y+\log c+\sigma^2 k)^2}{2\sigma^2}}dy$$
$$= \frac{1}{c}\sum_{k=0}^{\infty}(-1)^k e^{\frac{\sigma^2k^2}{2}+k\log c}\overline{\Phi}\left(\frac{\sigma^2 k+\log c}{\sigma}\right).$$

We can change the order of integration and summation in the second equation by the dominated convergence theorem since, for any $N = 0,1,2,\dots$ and $y \geq 0$,

$$\left|\sum_{n=0}^{N}(-1)^n e^{-ny}\right| = \frac{|1-(-e^{-y})^{N+1}|}{1+e^{-y}} \leq \frac{1+e^{-y(N+1)}}{1+e^{-y}} \leq 2.$$

Similarly, the third term (13) can be calculated as

$$(13) = \frac{1}{c}\sum_{k=0}^{\infty}(-1)^k e^{\frac{\sigma^2k^2}{2}-k\log c}\overline{\Phi}\left(\frac{\sigma^2 k-\log c}{\sigma}\right).$$

Substituting these into (11), (12) and (13), we obtain

$$E\left[\frac{1}{c+X}\right]$$
$$= \frac{1}{c}\overline{\Phi}\left(\frac{\log c}{\sigma}\right) - \frac{1}{c}\sum_{k=0}^{\infty}(-1)^k e^{\frac{\sigma^2k^2}{2}+k\log c}\overline{\Phi}\left(\frac{\sigma^2 k+\log c}{\sigma}\right)$$
$$+\frac{1}{c}\sum_{k=0}^{\infty}(-1)^k e^{\frac{\sigma^2k^2}{2}-k\log c}\overline{\Phi}\left(\frac{\sigma^2 k-\log c}{\sigma}\right)$$

and get the assertion of the proposition. //

B. *Using the ODE*

Proposition 5 can be proved by the ODE method.

Proof of Proposition 5. First, we calculate

$$\int_0^{\infty}\frac{1}{c+x}\exp\left(-\frac{x^2}{2\theta^2}\right)dx.$$

For $z > 0$, we define a function $g(z)$ as

$$g(z) = \int_0^{\infty}\frac{1}{z+x}\exp\left(-\frac{x^2}{2\theta^2}\right)dx.$$

Then, $g(z)$ satisfies a differential equation

$$g'(z) = -\frac{z}{\theta^2}g(z) - \frac{1}{z} + \sqrt{\frac{\pi}{2\theta^2}}.$$

This can be solved as follows:

$$\exp\left(\frac{c^2}{2\theta^2}\right)g(c)$$
$$= g(0) - \int_0^c \frac{1}{x}\exp\left(\frac{x^2}{2\theta^2}\right)dx + \sqrt{\frac{\pi}{2\theta^2}}\int_0^c \exp\left(\frac{x^2}{2\theta^2}\right)dx$$
$$= \frac{1}{2}\left(\int_0^{\infty}\frac{e^{-y}}{y}dy - \int_0^{\frac{c^2}{2\theta^2}}\frac{e^y}{y}dy\right) + \sqrt{\pi}\int_0^{\frac{c}{\sqrt{2\theta^2}}} e^{t^2}\,dt$$
$$= -\frac{1}{2}\left[\mathrm{Ei}\left(\frac{c^2}{2\theta^2}\right) - \pi\mathrm{i}\cdot\mathrm{erf}\left(-\frac{c}{\sqrt{2\theta^2}}\mathrm{i}\right)\right] \quad (14)$$

where the change of variables have been done as $y = x^2/(2\theta^2)$ and $t = x/\sqrt{2\theta^2}$. Then, the left hand side of (8) can be calculated as follows:

$$\mathrm{E}\left[\frac{1}{c+X}\right] = \frac{1}{\theta^2}\int_0^{\infty}\frac{x}{c+x}\exp\left(-\frac{x^2}{2\theta^2}\right)dx$$
$$= \frac{1}{\theta^2}\int_0^{\infty}\exp\left(-\frac{x^2}{2\theta^2}\right)dx - \frac{c}{\theta^2}\int_0^{\infty}\frac{1}{c+x}\exp\left(-\frac{x^2}{2\theta^2}\right)dx$$
$$= \sqrt{\frac{\pi}{2\theta^2}} + \frac{c}{2\theta^2}\exp\left(-\frac{c^2}{2\theta^2}\right)\left[\mathrm{Ei}\left(\frac{c^2}{2\theta^2}\right) - \pi\mathrm{i}\cdot\mathrm{erf}\left(-\frac{c}{\sqrt{2\theta^2}}\mathrm{i}\right)\right]$$

where we have used (14) in the last equation and this completes the proof. //

C. *Using the Mixed Distribution and the Laplace Transform*

Proposition 6 can be proved by introducing a mixed distribution and the Laplace transform.

Proof of Proposition 6. Let Z be a random variable such that $Z|X \sim \mathrm{Exp}(X)$, i.e., given X, Z follows the exponential distribution with parameter X. So, the conditional PDF of $Z|X$ is $f_{Z|X}(z|x) = xe^{-xz},\ z > 0,\ x > 0$. Then, the Laplace transform of Z can be calculated as

$$E[e^{-cZ}] = E\big[E[e^{-cZ}|X]\big] = E\left[\frac{X}{c+X}\right] = 1 - E\left[\frac{c}{c+X}\right],$$

for $c > 0$. Therefore, we get

$$E\left[\frac{1}{c+X}\right] = \frac{1-E[e^{cZ}]}{c}. \quad (15)$$

On the other hand, the unconditional PDF of Z is given as follows: for $z > 0$,

$$f_Z(z) = \int_0^{\infty} f_{Z|X}(z|x)f_X(x)\,dx$$
$$= \int_0^{\infty} xe^{-xz}\cdot\frac{\lambda^n}{\Gamma(n)}x^{n-1}e^{-\lambda x}\,dx$$
$$= \frac{\lambda^n}{\Gamma(n)}\int_0^{\infty} x^n e^{-(\lambda+z)x}\,dx$$
$$= \frac{n\lambda^n}{(\lambda+z)^{n+1}}.$$

Then, we have

$$E[e^{-cZ}] = \int_0^{\infty} e^{-cz}\frac{n\lambda^n}{(\lambda+z)^{n+1}}dz = ne^{c\lambda}\int_1^{\infty}\frac{e^{-c\lambda y}}{y^{n+1}}dy$$
$$= ne^{c\lambda}E_{n+1}(c\lambda),$$

where the change of variable in the second equation above has been done as $y = (\lambda+z)/\lambda$. Substituting this into (15), we obtain

$$E\left[\frac{1}{c+X}\right] = \frac{1-ne^{c\lambda}E_{n+1}(c\lambda)}{c} = \lambda e^{c\lambda}E_n(c\lambda),$$

since $nE_{n+1}(z) = e^{-z} - zE_n(z)$ (see eq.5.1.14, p.229 of [10]). //

Remark 1. Ref. [11] includes many other examples of analytical calculations of $E\,[1/(c\,+\,X)]$ when $0 < X < 1$. For instance, let X be a beta distributed random variable with parameters $(\alpha, 1-\alpha),\ 0 < \alpha < 1$. According to [11], $E[1/(1+\lambda\,X)]$ for $\lambda > 0$ can be obtained as follows. Suppose that Y is an exponential distributed random variable with parameter 1 and independent of X. Define $Z = XY$. Then, since Z follows the gamma distribution with parameters $(\alpha, 1)$,

$$\mathrm{E}\left[\frac{1}{1+\lambda X}\right] = E\left[e^{-\lambda XY}\right] = E\left[e^{-\lambda Z}\right] = \frac{1}{(1+\lambda)^{\alpha}}$$

(eq. (2.30), p.332 of [11]). See [11] for other examples.

IV. Conclusion

In this paper, the expectation of the reciprocal of first-degree polynomials of non-negative valued random variables $E[1/(c+X)]$ where $c > 0$ and X is a non-negative valued random variable is calculated. This is motivated to solve the maximization problem

$$\max_{0\le b\le 1} E[\log(1-b+bX)].$$

We have computed $E[1/(c+X)]$ when X is (i) the square of a Cauchy random variable, (ii) the absolute value of a Cauchy random variable, (iii) Pareto Type 2 distributed with integer shape parameters, (iv) log-normal distributed, (v) Rayleigh distributed and (vi) Erlang distributed. In an economic context, the solution of the above maximization is known as the Kelly criterion. Using the result of the computation of $E[1/(c+X)]$, the Kelly criterion can be obtained by finding b^* that satisfies

$$\mathrm{E}\left[\frac{-1+X}{1-b^*+b^*X}\right] = \frac{1}{b^*} - \frac{1}{(b^*)^2} E\left[\frac{1}{\frac{1-b^*}{b^*}+X}\right] = 0$$

or employing the Cover's algorithm.

We plan to extend this study to the multivariable case. When there is more than one risky asset, it is most effective to use the Cover algorithm: the Kelly criterion $b^* = (b_0^*, b_1^*, \dots, b_N^*)$ is obtained by $b^* = \lim_{n\to\infty} b^{(n)}$ where $b^{(n)} = \left(b_0^{(n)}, b_1^{(n)}, \dots, b_N^{(n)}\right)$, $n = 1,2,\dots$ is recurrently defined by

$$b_i^{(n+1)} = E\left[\frac{b_i^{(n)} X_i}{b_0^{(n)} + b_1^{(n)} X_1 + \cdots + b_N^{(n)} X_N}\right]$$

for $i = 1,2,\dots,N$, where $X_1, X_2, \dots, X_N$ denotes returns of N risky assets. In this case, the computation of the expectation

$$E\left[\frac{b_i^{(n)} X_i}{b_0^{(n)} + b_1^{(n)} X_1 + \cdots + b_N^{(n)} X_N}\right]$$

is critical.

Conflict of Interest

The authors declare no conflict of interest.

Acknowledgment

The author would like to express his gratitude to Prof. Takahiko Fujita of Chuo University for notifying him of the reference [11]. The author also thanks Mr. Shotaro Yagishita of Chuo University for his kind help in changing the order of the integration and the summation in the proof of Proposition 4.

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