On Solvents of Matrix Polynomials

Malika Yaici and Kamel Hariche

Abstract—This paper is concerned with the construction of right and left solvents (also called block roots) of a matrix polynomial from latent roots and vectors. It addresses also the important case of the existence of a complete set of block roots. Solvents do not always exist, so conditions for the existence of such solvents are discussed. The inverse of a matrix polynomial is obtained as a particular case of the block partial fraction expansion of a related rational matrix. It involves the knowledge of a complete set of solvents and the computation of the inverse of a block Vandermonde matrix. Numerical examples are given to illustrate the two results.

Index Terms—Block partial fraction expansion, latent roots, latent vectors, matrix fraction description, matrix polynomials, solvents.

I. INTRODUCTION

Matrix polynomials play a central role in the mathematical description of the dynamics of multivariable systems. In this case MIMO systems are described by a matrix transfer function $G(s)$ expressed as a right (or left) matrix fraction description (RMFD or LMFD):

$$\begin{cases} G(s) = N_r(s)D_r^{-1}(s) \\ \text{or} \\ G(s) = D_l^{-1}(s)N_l(s) \end{cases} \quad (1)$$

where N_r, D_r, N_l, D_l are matrix polynomials. This fact has led to an active research effort in matrix polynomials theory.

A theoretical introduction based on spectral approach of matrix polynomial theory is given in [1]; the same authors proposed an algorithm based on Jordan chains for the computation of a solvent of a matrix polynomial. Methods and algorithms for numerical solutions of spectral problems for one- and two-parameter polynomial and rational matrices are given in [2]. Some mathematical theory of λ -matrices, in particular solvents and interpolating λ -matrices are discussed in [3]. In this paper, the necessary and sufficient conditions for the existence of solvents and the corresponding generalized Lagrange interpolating λ -matrices are given.

Computing the inverse of a matrix polynomial is very important in all the fields dealing with matrix polynomials and has been treated by many researchers. Zhang [4] addressed the case of column reduced polynomial matrices to obtain an irreducible inverse. Schuster and Hippe [5] computed polynomial matrix inverses by interpolation, but the efficiency of the proposed algorithm is very dependent

on the interpolating points which are to be chosen.

An appropriate Sylvester resultant matrix has been used by Stefanidis *et al.*, [6] to generate a real matrix to compute its inverse. In [7] an algorithm to obtain a minimal state-space representation is given then Leverrier's algorithm is used to determine the transfer function representation of the inverse. In [8] the case of the generalized inversion of a polynomial matrix is treated and an algorithm to compute the inverse is proposed.

Many authors have considered using matrix polynomial s and block roots for solving some linear algebra problems or control problems such as: compensator design using numerical optimization techniques [9], block partial fraction expansion of a matrix fraction description (MFD) with single and repeated poles [10], reduction of the order of single-input single-output and multi-input multi-output robust controllers [11], feedforward compensator in two-degree-of-freedom controllers [12] etc.

In this paper, we will consider r -degree, m^{th} order monic matrix polynomials of the form

$$D(\lambda) = I_m \lambda^r + D_1 \lambda^{r-1} + \dots + D_{r-1} \lambda + D_r \quad (2)$$

where λ is a complex number and $D_i \in \mathbb{R}^{m \times m}$.

A latent root λ_i of $D(\lambda)$ is a complex number satisfying $\det D(\lambda_i) = 0$.

A right latent vector $X_i \in \mathbb{R}^m$ associated with λ_i satisfies $D(\lambda_i) X_i = \theta$ while a left latent vector Y_i is a row vector satisfying $Y_i D(\lambda_i) = \theta$ (θ being used here to indicate the zero vector).

A right solvent of $D(\lambda)$ is an $m \times m$ matrix R such that

$$R^r + D_1 R^{r-1} + \dots + D_{r-1} R + D_r = 0_m \quad (3)$$

while a left solvent is an $m \times m$ matrix L satisfying

$$L^r + L^{r-1} D_1 + \dots + L D_{r-1} + D_r = 0_m \quad (4)$$

The purpose of this paper is to show that right and left solvents can be constructed from latent roots and corresponding right and left latent vectors. It follows that each of the block roots contains a part of the latent structure of the matrix polynomial.

A particular case of interest is the one where we can construct a set of block roots covering all the latent structure of $D(\lambda)$. This set is referred to as a complete set of block roots. The existence of right and left solvents is discussed in the paper as well as the existence of a complete set. Furthermore, the inverse $D^{-1}(\lambda)$ is given as a particular case of the block partial fraction expansion of a related rational matrix, this result being the main novelty of the paper.

The paper is organized as follows: after an introduction to the most important facts on matrix polynomials and lambda-matrices, a section is dedicated to the construction of

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M. Yaici is with the Computer Department, University of Bejaia, Bejaia 06000, Algeria (e-mail: yaici_m@hotmail.com).

K. Hariche is with Electronics and Electronics Institute, University of Boumerdes, Boumerdes 35000, Algeria (e-mail: khariche@yahoo.com).

right/left solvents from latent roots and latent vectors along with the condition of their existence. A section dealing with the block partial fraction expansion of matrix polynomials is given followed by the determination of $D^{-1}(\lambda)$. Finally a conclusion and some perspectives finish the paper.

II. THEORETICAL PRELIMINARIES

An m^{th} order r^{th} degree λ -matrix is given by:

$$D(\lambda) = D_0\lambda^r + D_1\lambda^{r-1} + \dots + D_{r-1}\lambda + D_r \quad (5)$$

where D_i ($i = 0 \dots r$) are $m \times m$ complex matrices and λ a complex number.

For an $m \times m$ complex matrix X we define the following:

A right polynomial matrix is given by:

$$D_R(X) = D_0X^r + D_1X^{r-1} + \dots + D_{r-1}X + D_r \quad (6)$$

A left polynomial matrix is given by:

$$D_L(X) = X^rD_0 + X^{r-1}D_1 + \dots + XD_{r-1} + D_r \quad (7)$$

If D_0 is an identity matrix I_m then the matrix polynomial is said to be monic.

If D_0 is singular and $\det D(\lambda) \neq 0$ for any λ then $D(\lambda)$ can be reversed to make D_0 non-singular.

A. Latent Roots and Latent Vectors

A latent root λ_i of $D(\lambda)$ is a complex number satisfying $\det D(\lambda_i) = 0$; A right latent vector $X_i \in \mathbb{R}^m$ associated with λ_i satisfies $D(\lambda_i)X_i = \theta$; while a left latent vector Y_i is a row vector satisfying $Y_iD(\lambda_i) = \theta$.

If $D(\lambda)$ has a singular leading coefficient (D_0) then $D(\lambda)$ has latent roots at infinity.

B. Solvents

Right and left solvents are defined in equations 3 and 4. The following are important facts on Solvents [14]:

- 1) Solvents of a λ -matrix do not always exist.
- 2) If R is a right solvent then there exists a λ -matrix Q of degree $r-1$ such that $D(\lambda) = Q(\lambda)(I\lambda - R)$.
- 3) If L is a left solvent then there exists a λ -matrix S of degree $r-1$ such that $D(\lambda) = (I\lambda - L)S(\lambda)$.
- 4) Generalized right (left) eigenvectors of a right (left) solvent are the generalized latent vectors of $D(\lambda)$.
- 5) W is a weak solvent if $D(W)$ is singular.

C. Construction of Matrix Coefficients of a λ -Matrix

A polynomial matrix can be determined (or its matrix coefficients) from a set of solvents under some conditions [9]

For each right solvent R_j ($j=1 \dots r$) of $D(\lambda)$ we have: $R_j^r + D_1R_j^{r-1} + \dots + D_{r-1}R_j + D_r = 0_m$, then $D_1R_j^{r-1} + \dots + D_{r-1}R_j + D_r = -R_j^r$ or for $j = 1 \dots r$.

$$[D_r \quad D_{r-1} \quad \dots \quad D_1] \begin{bmatrix} I_m \\ R_j \\ \vdots \\ R_j^{r-1} \end{bmatrix} = -R_j^r$$

Then:

$$[D_r \quad D_{r-1} \quad \dots \quad D_1] = -[R_1^r \quad R_2^r \quad \dots \quad R_r^r]V_R^{-1} \quad (8)$$

where V_R is a "right" block Vandermonde matrix:

$$V_R = \begin{bmatrix} I_m & I_m & \dots & I_m \\ R_1 & R_2 & \dots & R_r \\ \vdots & \vdots & \dots & \vdots \\ R_1^{r-1} & R_2^{r-1} & \dots & R_r^{r-1} \end{bmatrix} \quad (9)$$

For each left solvent L_j ($j=1 \dots r$) of $D(\lambda)$ we have:

$$L_j^r + L_j^{r-1}D_1 + \dots + L_jD_{r-1} + D_r = 0_m$$

Then following the same development:

$$\begin{bmatrix} D_r \\ D_{r-1} \\ \vdots \\ D_1 \end{bmatrix} = -V_L^{-1} \begin{bmatrix} L_1^r \\ L_2^r \\ \vdots \\ L_r^r \end{bmatrix} \quad (10)$$

where V_L is a block Vandermonde matrix:

$$V_L = \begin{bmatrix} I_m & L_1 & \dots & L_1^{r-1} \\ I_m & L_2 & \dots & L_2^{r-1} \\ \vdots & \vdots & \dots & \vdots \\ I_m & L_r & \dots & L_r^{r-1} \end{bmatrix} \quad (11)$$

Theorem 1: A block **Vandermonde** matrix constructed from a complete set of solvents of a particular matrix polynomial is non singular

Proof 1: see [13].

III. CONSTRUCTION OF SOLVENTS

A. Construction of Right Solvents

Suppose that the set $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ of m latent roots of $D(\lambda)$ has a linearly independent set of corresponding right latent vectors $\{X_1, X_2, \dots, X_m\}$. Let $M = (X_1 \ X_2 \ \dots \ X_m)$ be the $m \times m$ matrix whose columns are the linearly independent right

latent vectors and $M^{-1} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{pmatrix}$ be its inverse.

It follows that $\sum_{i=1}^m X_i Y_i = I_m$ and $X_i Y_j = \delta_{ij}$ for $i, j = 1 \dots m$ (from $MM^{-1} = M^{-1}M = I_m$).

Theorem 2: The $m \times m$ matrix $R = MAM^{-1}$, where $A = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$, is a right solvent of $D(\lambda)$.

Proof 2: Let $D_R(R) = R^r + D_1R^{r-1} + \dots + D_{r-1}R + D_r$

Hence $D_R(R)$ can be rewritten as:

$$D(R) = M\Lambda^r M^{-1} + D_1M\Lambda^{r-1}M^{-1} + \dots + D_{r-1}M\Lambda M^{-1} + D_r \quad (12)$$

It can easily be verified that $M\Lambda^l M^{-1} = \sum_{i=1}^m \lambda_i^l X_i Y_i = I$ for $l = 1 \dots r$ which gives

$$D(R) = \sum_{i=1}^m \lambda_i^r X_i Y_i + D_1 \sum_{i=1}^m \lambda_i^{r-1} X_i Y_i + \dots + D_{r-1} \sum_{i=1}^m \lambda_i X_i Y_i + D_r \quad (13)$$

or

$$D(R) = (\lambda_1^r X_1 + \lambda_1^{r-1} D_1 X_1 + \dots + \lambda_1 D_{r-1} X_1) Y_1 + (\lambda_2^r X_2 + \lambda_2^{r-1} D_1 X_2 + \dots + \lambda_2 D_{r-1} X_2) Y_2 + \dots + (\lambda_m^r X_m + \lambda_m^{r-1} D_1 X_m + \dots + \lambda_m D_{r-1} X_m) Y_m + D_r \quad (14)$$

Since $\{\lambda_i, X_i\}$ is a latent pair it follows that:

$$\lambda_i^r X_i + \lambda_i^{r-1} D_1 X_i + \dots + \lambda_i D_{r-1} X_i + D_r X_i = \theta \quad (15)$$

Or

$$\lambda_i^r X_i + \lambda_i^{r-1} D_1 X_i + \dots + \lambda_i D_{r-1} X_i = -D_r X_i \quad (16)$$

Leading to

$$D(R) = -D_r X_1 Y_1 - D_r X_2 Y_2 - \dots - D_r X_m Y_m + D_r \\ = -D_r I_m + D_r = 0_m \quad (17)$$

Hence showing that a matrix R constructed from a set of m latent roots and a corresponding set of m linearly independent right latent vectors is a right solvent of the matrix polynomial $D(\lambda)$.

It can easily be verified that the right latent vector X_i of $D(\lambda)$ associated with the latent root λ_i is a right eigenvector of the right solvent R . Since λ_i is an eigenvalue of R , it follows that the right eigenstructure of R captures a part of the right latent structure of $D(\lambda)$.

B. Condition of Existence

From the above construction, the nodal point is the existence of a set of m linearly independent right latent vectors associated with a set of m latent roots of $D(\lambda)$. Existence of such a set insures the existence of a right solvent. If we let $\sigma[D(\lambda)]$ denote the set of all latent roots of $D(\lambda)$ and $\sigma[R_i]$ the set of eigenvalues of the right solvent R_i , then a complete set of right solvents is obtained if we can find r right solvents such that:

$$\begin{cases} \bigcup_{i=1}^r \sigma[R_i] = \sigma[D(\lambda)] \\ \sigma[R_i] \cap \sigma[R_j] = \emptyset \end{cases} \quad (18)$$

A complete set of right solvents will then describe completely the latent structure of $D(\lambda)$.

C. Construction of Left Solvents

In a similar manner, we will establish that a left solvent $L \in \mathbb{R}^{m \times m}$ can be constructed from a set $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ of m latent roots and a corresponding set of m linearly independent left latent (row) vectors $\{Y_1, Y_2, \dots, Y_m\}$.

Theorem 3: The $m \times m$ matrix $L = P^{-1}AP$, where

$$P = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{pmatrix}, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_m) \quad \text{and} \quad P^{-1} =$$

$(X_1 \ X_2 \ \dots \ X_m)$ is a left solvent of $D(\lambda)$.

Proof 3: we have $\sum_{i=1}^m X_i Y_i = I_m$ and $X_i Y_j = \delta_{ij}$ for $i, j = 1 \dots m$.

Evaluating $D(L) = L^r + L^{r-1}D_1 + \dots + LD_{r-1} + D_r$ leads to

$$D(L) = P^{-1}AP^r + P^{-1}A^{r-1}PD_1 + \dots + P^{-1}APD_{r-1} + D_r \quad (19)$$

or

$$D(L) = \sum_{i=1}^m \lambda_i^r X_i Y_i + (\sum_{i=1}^m \lambda_i^{r-1} X_i Y_i) D_1 \\ + \dots + (\sum_{i=1}^m \lambda_i X_i Y_i) D_{r-1} + D_r \quad (20)$$

or

$$D(L) = X_1 [\lambda_1^r Y_1 + \lambda_1^{r-1} Y_1 D_1 + \dots + \lambda_1 Y_1 D_{r-1}] + \\ X_2 [\lambda_2^r Y_2 + \lambda_2^{r-1} Y_2 D_1 + \dots + \lambda_2 Y_2 D_{r-1}] + \dots + \\ X_m [\lambda_m^r Y_m + \lambda_m^{r-1} Y_m D_1 + \dots + \lambda_m Y_m D_{r-1}] + D_r \quad (21)$$

From $Y_i D(\lambda_i) = \theta$ ($i = 1 \dots m$) it follows that

$$\lambda_i^r Y_i + \lambda_i^{r-1} Y_i D_1 + \dots + \lambda_i Y_i D_{r-1} = -Y_i D_r \quad (22)$$

and

$$D(L) = -X_1 Y_1 D_r - X_2 Y_2 D_r - \dots - X_m Y_m D_r \\ = -I_m D_r + D_r = 0_m \quad (23)$$

Indicating that L is indeed a left solvent of $D(\lambda)$.

Here again, we can verify that the left latent vector of $D(\lambda)$ associated with the latent root λ_i is a left eigenvector of the left solvent L . Since λ_i is an eigenvalue of L , it follows that the left eigenstructure of L captures a part of the left latent structure of $D(\lambda)$.

D. Condition of Existence

Just as for the right solvent, the existence of a left solvent depends on the existence of a set of m linearly independent left latent vectors. A complete set of left solvents (covering totally the latent structure of $D(\lambda)$) is obtained if we can find r left solvents where each solvent involves a distinct set of m latent roots of $D(\lambda)$. This in turn requires that for each such a distinct set, we can find a corresponding set of linearly left latent vectors.

E. Illustrative Example

Consider

$$D(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \lambda^3 + \begin{pmatrix} 0 & 1 \\ 0 & 5 \end{pmatrix} \lambda^2 \\ + \begin{pmatrix} -1 & 5 \\ 0 & 6 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} \quad (24)$$

where the latent roots are $\{0, 1, -1, -2, -3\}$ with 0 being a double root. The right latent vectors corresponding to these latent roots are respectively:

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -12 \end{pmatrix} \right\} \quad (25)$$

Each right solvent being a 2×2 matrix, we form right solvents by pairing latent roots with corresponding linearly independent right latent vectors such as $\{0, -2\}$, $\{0, -3\}$, $\{1, -2\}$, $\{1, -3\}$, $\{-1, -2\}$, $\{-1, -3\}$ or $\{-2, -3\}$.

The right solvent, say involving the pair $\{1, -2\}$ is then

$$R = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \quad (26)$$

It can be checked that $R^3 + D_1 R^2 + D_2 R + D_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

We cannot have in this case a complete set of right solvents since out of the seven possible right solvents, we cannot obtain three disjoint spectra covering the whole spectrum of $D(\lambda)$.

Left latent vectors associated with the latent roots are

$$\{(0 \ 1), (-6 \ 5), (1 \ 0), (0 \ 1), (0 \ 1)\} \quad (27)$$

The left solvent involving, say, the latent root $\{1, -2\}$ is then:

$$L = \begin{pmatrix} -6 & 5 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -6 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2.5 \\ 0 & -2 \end{pmatrix} \quad (28)$$

IV. EXPANSION OF $D^{-1}(s)$

In this section we will show that we can determine the inverse of a matrix polynomial $D(\lambda)$ from the block partial fraction expansion of the transfer function of a system described by a right or left matrix fraction description (1).

A. Left MFD

Consider the matrix transfer function of the system given in left matrix fraction description:

$$G(s) = [Is^r + D_1s^{r-1} + \dots + D_{r-1}s + D_r]^{-1} [N_0s^{r-1} + N_1s^{r-2} + \dots + N_{r-1}] \quad (29)$$

And $\{L_1, L_2, \dots, L_r\}$ a complete set of left solvents of $D(s)$. It can be verified that the state space equation of this system in block observer form is the following:

$$\begin{cases} \dot{x}_o = A_o x_o + B_o u \\ y = C_o x_o \end{cases} \quad (30)$$

where

$$\begin{cases} A_o = \begin{pmatrix} 0_m & \dots & 0_m & -D_r \\ I_m & \dots & 0_m & -D_{r-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0_m & \dots & I_m & -D_1 \end{pmatrix} \\ B_o = \begin{pmatrix} N_{r-1} \\ N_{r-2} \\ \vdots \\ N_0 \end{pmatrix} \\ C_o = (0_m \quad \dots \quad 0_m \quad I_m) \end{cases} \quad (31)$$

satisfies

$$G(s) = C_o(sI - A_o)^{-1} B_o \quad (32)$$

Using the transformation $x_d = V_L x_o$ with V_L is the left block Vandermonde matrix, which is assumed non-singular, we obtain:

$$\begin{cases} \dot{x}_d = A_d x_d + B_d u \\ y = C_d x_d \end{cases} \quad (33)$$

$$\text{where } \begin{cases} A_d = V_L A_o V_L^{-1} \\ B_d = V_L B_o \\ C_d = C_o V_L^{-1} \end{cases} \text{ with } \begin{cases} A_d = \begin{pmatrix} L_1 & \cdot & \cdot & \cdot \\ \cdot & L_2 & (0) & \cdot \\ \cdot & (0) & \ddots & \cdot \\ \cdot & \cdot & \cdot & L_r \end{pmatrix} \\ B_d = \begin{pmatrix} B_{d1} \\ B_{d2} \\ \vdots \\ B_{dr} \end{pmatrix} = \begin{pmatrix} N_{r-1} + L_1 N_{r-2} + \dots + L_1^{r-1} N_0 \\ N_{r-1} + L_2 N_{r-2} + \dots + L_2^{r-1} N_0 \\ \vdots \\ N_{r-1} + L_r N_{r-2} + \dots + L_r^{r-1} N_0 \end{pmatrix} \\ C_d = (C_{d1} \quad C_{d2} \quad \dots \quad C_{dr}) \end{cases} \quad (34)$$

C_L is the last block row of V_L^{-1} .

Now

$$G(s) = C_d(sI - A_d)^{-1} B_d = \sum_{i=1}^r C_{di}(sI - L_i)^{-1} B_{di} \quad (35)$$

Hence a block partial fraction expansion of $G(s)$ about a complete set of left solvents is obtained.

A particular case of interest is obtained if: $N_0 = N_1 = \dots = N_{r-2} = 0$ and $N_{r-1} = I$, in which case we obtain the decomposition of $D^{-1}(s)$.

$$\text{Since } B_o = \begin{pmatrix} I_m \\ 0_m \\ \vdots \\ 0_m \end{pmatrix}$$

Then

$$B_d = V_L B_o = \begin{pmatrix} I_m & L_1 & \dots & L_1^{r-1} \\ I_m & L_2 & \dots & L_2^{r-1} \\ \vdots & \vdots & \ddots & \vdots \\ I_m & L_r & \dots & L_r^{r-1} \end{pmatrix} \begin{pmatrix} I_m \\ 0_m \\ \vdots \\ 0_m \end{pmatrix} = \begin{pmatrix} I_m \\ I_m \\ \vdots \\ I_m \end{pmatrix}$$

and C_L is the last block row of V_L^{-1} .

Hence

$$D^{-1}(s) = \sum_{i=1}^r C_{Li}(sI - L_i)^{-1} \quad (36)$$

Theorem 4: The inverse of a matrix polynomial $D(\lambda)$ is given by (36), if $D(\lambda)$ presents a complete set of left solvents.

Proof 4: the non-singularity of the left block Vandermonde matrix V_L is verified if a complete set of left solvents exist.

B. Right MFD

In a similar manner we can determine $D^{-1}(s)$ from a right MFD of a matrix transfer function. We consider a matrix transfer function as follows:

$$G(s) = [N_0s^{r-1} + N_1s^{r-2} + \dots + N_{r-1}] [Is^r + D_1s^{r-1} + \dots + D_{r-1}s + D_r]^{-1} \quad (37)$$

With $\{R_1, R_2, \dots, R_r\}$ a complete set of right solvents of $D(s)$.

The corresponding state space equation in block controller form is:

$$\begin{cases} \dot{x}_c = \begin{pmatrix} 0_m & I_m & \dots & 0_m \\ \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & \dots & I_m \\ -D_r & -D_{r-1} & \dots & -D_1 \end{pmatrix} x_c + \begin{pmatrix} 0_m \\ \vdots \\ 0_m \\ I_m \end{pmatrix} u \\ y = (N_{r-1} \quad N_{r-2} \quad \dots \quad N_1 \quad N_0) x_c \end{cases} \quad (38)$$

Satisfies $(s) = C_c(sI - A_c)^{-1} B_c$.

Using the transformation $x_c = V_R x_d$ with V_R is an assumed non-singular "right" block Vandermonde matrix, we obtain:

$$\begin{cases} \dot{x}_d = A_d x_d + B_d u \\ y = C_d x_d \end{cases} \quad (39)$$

$$\text{where } \begin{cases} A_d = V_R^{-1} A_c V_R \\ B_d = V_R^{-1} B_c \\ C_d = C_c V_R \end{cases} \text{ with}$$

$$\begin{cases} A_d = \begin{pmatrix} R_1 & \cdot & \cdot & \cdot \\ \cdot & R_2 & (0) & \cdot \\ \cdot & (0) & \ddots & \cdot \\ \cdot & \cdot & \cdot & R_r \end{pmatrix}, B_d = \begin{pmatrix} B_{d1} \\ B_{d2} \\ \vdots \\ B_{dr} \end{pmatrix} \\ C_d = \begin{pmatrix} C_{d1} \\ C_{d2} \\ \vdots \\ C_{dr} \end{pmatrix}^T = \begin{pmatrix} N_{r-1} + N_{r-2}R_1 + \dots + N_0R_1^{r-1} \\ N_{r-1} + N_{r-2}R_2 + \dots + N_0R_2^{r-1} \\ \vdots \\ N_{r-1} + N_{r-2}R_r + \dots + N_0R_r^{r-1} \end{pmatrix}^T \end{cases} \quad (40)$$

B_d is the last block column of V_R^{-1} .

It follows that $G(s) = C_R(sI - A_R)^{-1}B_R = \sum_{i=1}^r C_{R_i}(sI - R_i)^{-1}B_{R_i}$

Leading to a block partial fraction expansion of $G(s)$ about a complete set of right solvents.

If $N_0 = N_1 = \dots = N_{r-2} = 0$ and $N_{r-1} = I$, the decomposition of $D^{-1}(s)$ follows.

Since $C_c = (I_m \ 0_m \ \dots \ 0_m)$ and V_R is the right Vandermonde matrix (as given in (9)), then $C_R = C_c V_R = (I_m \ \dots \ I_m)$

And B_R is the last block column of V_R^{-1} .

It follows that:

$$D(s) = \sum_{i=1}^r (sI - R_i)^{-1}B_{R_i} \quad (41)$$

Theorem 5: The inverse of a matrix polynomial $D(\lambda)$ is given by (41) iff $D(\lambda)$ presents a complete set of right solvents.

Proof 5: The condition for the previous development to be possible is the fact that the right block Vandermonde matrix V_R is non-singular, and this is verified if a complete set of right solvents exist.

C. Illustrative Example

Let

$$D(s) = Is^2 + \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}s + \begin{pmatrix} 0 & -1 \\ 0 & -2 \end{pmatrix} \quad (42)$$

with the following left solvents:

$$L_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \quad (43)$$

Hence

$$V_L = \begin{pmatrix} I & L_1 \\ I & L_2 \end{pmatrix} \Rightarrow V_L^{-1} = \begin{pmatrix} 0 & 1/3 & 1 & -1/3 \\ 0 & 2/3 & 0 & 1/3 \\ 1 & -1/3 & -1 & 1/3 \\ 0 & -1/3 & 0 & 1/3 \end{pmatrix}$$

it follows that:

$$D^{-1}(s) = C_{L_1}(sI - L_1)^{-1} + C_{L_2}(sI - L_2)^{-1} \quad (44)$$

$$D(s) = \begin{pmatrix} 1 & -1/3 \\ 0 & -1/3 \end{pmatrix} \begin{pmatrix} s-1 & 0 \\ 0 & s+1 \end{pmatrix}^{-1} + \begin{pmatrix} -1 & 1/3 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} s & -1 \\ 0 & s-2 \end{pmatrix}^{-1}$$

$$\text{Then } D^{-1}(s) = \begin{pmatrix} \frac{1}{s(s-1)} & \frac{-1}{s(s+1)(s-2)} \\ 0 & \frac{1}{(s+1)(s-2)} \end{pmatrix} \quad (45)$$

V. CONCLUSION

Solvents or block roots of a matrix polynomial can be constructed from latent roots and latent vectors. Conditions of their existence are discussed. The block partial fraction expansion of a matrix transfer function expressed as a matrix fraction description is obtained. From this expansion, the inverse of a matrix polynomial is determined and it involves the knowledge of a complete set of solvents and the

computation of the inverse of a block Vandermonde matrix.

In control theory, dealing with systems represented in matrix fraction description requires often a full set of solvents, and for the case of the block Vandermonde matrix inversion, a recursive algorithm has been elaborated. The importance of solvents or block roots (blocks poles and block zeros) in control systems design stems from the fact that block pole assignment is equivalent to eigenstructure assignment (eigenvalues and eigenvectors) not merely the assignment of poles. This research work is a part of a thesis work on the design of compensators for systems described by matrix fraction descriptions.

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Malika Yaici is a PhD student at the Institute of Electronics and Electro-techniques of the University of Boumerdes, Algeria. She works on compensators design for MIMO systems. She is also an associate lecturer at the Computer Department of the University of Bejaia, Algeria, where she teaches parallel and distributed algorithms, operating systems and network specification. Her major fields includes distributed systems, control theory, distributed applications, matrix algebra. The paper is part of the thesis research work under the direction of Professor Kamel Hariche.

Kamel Hariche received a B.S. degree in physics from the University of Algiers, Algeria (1973), a master's degree in 1978. And a PhD degree in electrical engineering from the University of Houston (Texas-USA). Currently, he is a professor in electrical engineering at the Institute of Electronics and Electro-techniques of the University of Boumerdes, Algeria. He wrote numerous papers in control systems and applied mathematics.