Full-Order State Observer Design for Nonlinear Systems Based on Piecewise Bilinear Models

Tadanari Taniguchi, Luka Eciolaza, and Michio Sugeno

***Abstract*—This paper proposes a full-order state observer design of nonlinear control systems approximated by piecewise bilinear (PB) models. We construct PB model of nonlinear control systems. The approximated system is found to be fully parametric. The input-output (I/O) feedback linearization is applied to stabilize PB control systems. The design method is capable of designing the state observer and the servo controller of nonlinear systems separately. Although the controller is simpler than the conventional I/O feedback linearization controller, the control performance based on PB model is the same as the conventional one. We present the PB modeling combined with the conventional feedback linearization as a very powerful tool for the analysis and synthesis of nonlinear control systems. An illustrative example confirms the feasibility of our proposals.**

***Index Terms*—Nonlinear control, piecewise bilinear model, input-output linearization, state observer.**

I. Introduction

Piecewise linear (PL) systems which are fully parametric have been intensively studied in connection with nonlinear systems [1]-[4]. We are interested in the parametric piecewise approximation of nonlinear control systems based on the original idea of PL approximation. The PL approximation has general approximation capability for nonlinear functions with a given precision. However, it is difficult to handle some PL system such as simplexes in the rectangular coordinate system.

To overcome this difficulty, one of the authors suggested to use the piecewise bilinear (PB) approximation [5]. We note that a bilinear function as a basis of PB approximation is, as a nonlinear function, the second simplest one after a linear function. The model has the following features. 1) The PB model is derived from fuzzy if-then rules with singleton consequents. 2) It is built on piecewise hypercubes partitioned in the state space. 3) It has general approximation capability for nonlinear systems. 4) It is a piecewise nonlinear model, the second simplest after a PL model. 5) It is continuous and fully parametric. So far we have shown the necessary and sufficient conditions for the stability of PB systems with respect to Lyapunov functions in the two dimensional case [6], [7] where membership functions are fully taken into account. We derived the stabilizing conditions [8], [9] based on the feedback linearization, where [8] applies the input-output linearization and [9]-[10] applies the full-state linearization. In the feedback linearization, we design a state feedback controller which transforms a nonlinear system into an equivalent linear system.

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This paper proposes a full-order state observer of PB control system to estimate the coordinate transformation z. This design method is capable of designing the state observer and the controller of nonlinear systems separately. The full-order state observer can be designed for all the PB control systems since the feedback linearized system is observable. Although the controller and observer are simpler than the conventional I/O feedback linearization controller, the performance based on PB model is equivalent to the conventional one.

This paper is organized as follows. Section II presents the canonical form of PB models. Section III presents PB controllers for nonlinear plants with PB modeling and I/O linearization. Section IV proposes a design method of observer-based PB controller. Section V shows an example demonstrating the feasibility of the proposed methods, and Section VI gives conclusions.

II. Canonical Form of PB Models

A. Open-Loop Systems

In this section, we introduce the PB models suggested in [5]. We deal with the two dimensional case without loss of generality. Define a vector $d(\sigma,\tau)$ and a rectangle $R_{\sigma\tau}$ in the two-dimensional space as, respectively,

$$\begin{aligned} d(\sigma,\tau) &\equiv \left(d_1(\sigma), d_2(\tau)\right)^T \\ R_{\sigma\tau} &\equiv [d_1(\sigma), d_1(\sigma+1)]\times[d_2(\tau), d_2(\tau+1)] \end{aligned} \quad (1)$$

σ and τ are integers: $-\infty<\sigma,\tau<\infty$ where $d_1(\sigma)<d_1(\sigma+1)$, $d_2(\tau)<d_2(\tau+1)$ and $d(0,0)\equiv(d_1(0),d_2(0))^T$. The superscript T denotes transpose operation. For $x\in R_{\sigma\tau}$, the PB system is expressed as

$$\begin{cases} \dot{x} = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2)\, f(i,j), \\ x = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) d(i,j), \end{cases} \quad (2)$$

where

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$$\begin{cases} \omega_1^{\sigma}(x_1) = (d_1(\sigma+1) - x_1) / (d_1(\sigma+1) - d_1(\sigma)), \\ \omega_1^{\sigma+1}(x_1) = (x_1 - d_1(\sigma)) / (d_1(\sigma+1) - d_1(\sigma)), \\ \omega_2^{\tau}(x_2) = (d_2(\tau+1) - x_2) / (d_2(\tau+1) - d_2(\tau)), \\ \omega_2^{\tau+1}(x_2) = (x_2 - d_2(\tau)) / (d_2(\tau+1) - d_2(\tau)), \end{cases} \tag{3}$$

and $\omega_2^j(x_2) \in [0,1]$. In the above, we assume $f(0,0) = 0$ and $d(0,0) = 0$ to guarantee $\dot{x} = 0$ for $x = 0$.

A key point in the system is that the state variable x is also expressed by a convex combination of $d(i,j)$ with respect to $\omega_1^i(x_1)$ and $\omega_2^j(x_2)$ just as in the case of $\dot{x}$. As is seen in Eq. (3), x is located inside $R_{\sigma\tau}$ which is a rectangle: a hypercube in general. That is, the expression of x is polytopic with four vertices $d_1(\sigma)$, $d_1(\sigma+1)$, $d_2(\tau)$ and $d_2(\tau+1)$. The model of $\dot{x} = f(x)$ is built on a rectangle including x in the state space and it is also poly topic with four vertices $f(\sigma,\tau)$, $f(\sigma+1,\tau)$, $f(\sigma,\tau+1)$ and $f(\sigma+1,\tau+1)$ .We call this form of the canonical model (2) parametric expression.

Representing $\dot{x}$ with x in Eqs. (2) and (3), we can obtain the state space expression of the model which is found to be bilinear (bi-affine) [5]. Therefore, the derived PB model has simple nonlinearity. In the case of the PL approximation, a PL model is built on simplexes partitioned in the state space, triangles in the two dimensional case. Note that any three points in the three dimensional space are spanned with an affine plane: $y = a + bx_1 + cx_2$. A PL model is continuous. It is, however, difficult to handle simplexes in the rectangular coordinate system.

Also we can see that any four points in the three dimensional space can be spanned with a bi-affine plane: $y = a + bx_1 + cx_2 + dx_1x_2$. In contract to a PL model, a PB model as such is built on rectangles with the four vertices, on hypercubes in a general dimensional space, partitioned in the state space; it well matches the rectangular coordinate system. Therefore, PB models would be applicable to control purpose.

B. *Closed-Loop Systems*

We consider a two-dimensional nonlinear control system.

$$\begin{cases} \dot{x} = f_o(x) + g_o(x)u(x) \\ y = h_o(x). \end{cases} \tag{4}$$

The PB model (5) can be constructed from the nonlinear system (4).

$$\begin{cases} \dot{x} = f(x) + g(x)u(x), \\ y = h(x), \end{cases} \tag{5}$$

where

$$\begin{cases} f(x) = \sum_{i=\sigma}^{\sigma+1} \sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) f(i,j), \\ g(x) = \sum_{i=\sigma}^{\sigma+1} \sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) g(i,j), \\ h(x) = \sum_{i=\sigma}^{\sigma+1} \sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) h(i,j). \end{cases} \tag{6}$$

The modeling procedure in the region $R_{\sigma\tau}$ is as follows.

Algorithm 2.1: Piecewise bilinear modeling procedure

1) Assign vertices $d(i,j)$ for $x_1 = d_1(\sigma)$, $d_1(\sigma+1)$, $x_2 = d_2(\tau)$, $d_2(\tau+1)$ of the state vector x, then the state space is partitioned into piecewise regions, see also Fig. 1.
2) Compute the vertices $f(i,j)$, $g(i,j)$ and $h(i,j)$ in Eqs. (6), by substituting the values of $x_1 = d_1(\sigma)$, $d_1(\sigma+1)$ and $x_2 = d_2(\tau)$, $d_2(\tau+1)$ into original nonlinear functions f_o, g_o and h_o in the system (4). Fig. 1 illustrates $f_1(x)$ and $x \in R_{\sigma\tau}$.

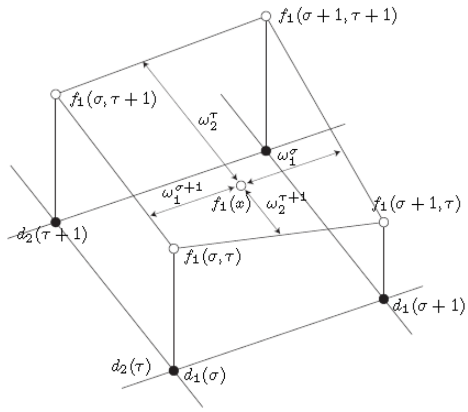

Fig. 1. Piecewise region ($f_1(x)$, $x \in R_{\sigma\tau}$).

The overall PB model can be obtained automatically when all the vertices are assigned. Note that $f(x)$, $g(x)$ and $h(x)$ in the PB model coincide with those in the original system at the vertices of all the regions.

III. Design of PB Controllers for Nonlinear Models with PB Modeling and I/O Linearization

This section deals with the I/O linearization of nonlinear control systems approximated with PB models. We consider, in particular, nonlinear systems and show their I/O linearization based on PB models in detail. We also show that in the case of PB systems, the I/O linearization (the feedback linearization in general) may be applicable to a global region by avoiding the restrictions of the conventional linearization of nonlinear control system: the restriction concerning the relative degree. First we give a brief introduction to the I/O linearization of PB models [9], [10]. Due to lack of space, we use $\omega_1^{i_1}$ and $\omega_2^{i_2}$ in $\omega_1^{i_1}(x_1)$ and $\omega_2^{i_2}(x_2)$ in this section.

A. *I/O linearization*

Consider the PB model (5) in the previous section. The derivative y is given by

$$\dot{y} = \frac{\partial h}{\partial x}\big(f(x) + g(x)u\big) = L_f h(x) + L_g h(x)u,$$

where

$$L_f h(x)=\sum_{i_2=\sigma_2}^{\sigma_2+1}\omega_2^{i_2}\cdots\sum_{i_n=\sigma_n}^{\sigma_n+1}\omega_n^{i_n}\frac{h(\delta_1,i_2,\ldots,i_n)}{d_1(\delta_1)}f_1+\cdots+\sum_{i_1=\sigma_1}^{\sigma_1+1}\omega_1^{i_1}\cdots\sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1}\omega_{n-1}^{i_{n-1}}\frac{h(i_1,i_2,\ldots,\delta_n)}{d_n(\delta_n)}f_n,$$

$$L_g h(x)=\sum_{i_2=\sigma_2}^{\sigma_2+1}\omega_2^{i_2}\cdots\sum_{i_n=\sigma_n}^{\sigma_n+1}\omega_n^{i_n}\frac{h(\delta_1,i_2,\ldots,i_n)}{d_1(\delta_1)}g_1+\cdots+\sum_{i_1=\sigma_1}^{\sigma_1+1}\omega_1^{i_1}\cdots\sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1}\omega_{n-1}^{i_{n-1}}\frac{h(i_1,i_2,\ldots,\delta_n)}{d_n(\delta_n)}g_n,$$

$$\begin{aligned}
h(\delta_1,i_2,\ldots,i_n)&=h(\sigma_1+1,i_2,\ldots,i_n)-h(\sigma_1,i_2,\ldots,i_n),\\
h(i_1,\delta_2,\ldots,i_n)&=h(i_1,\sigma_2+1,\ldots,i_n)-h(i_1,\sigma_2,\ldots,i_n),\\
h(i_1,i_2,\ldots,\delta_n)&=h(i_1,i_2,\ldots,\sigma_n+1)-h(i_1,i_2,\ldots,\sigma_n),\\
d_1(\delta_1)&=d_1(\sigma_1+1)-d_1(\sigma_1),\\
d_2(\delta_2)&=d_2(\sigma_2+1)-d_2(\sigma_2),\\
d_n(\delta_n)&=d_n(\sigma_n+1)-d_n(\sigma_n).
\end{aligned}$$

If $L_g h(x)=0$, then $\dot{y}=L_f h(x)$ is independent of u. We continue to calculate the second derivative of y, denoted by $y^{(2)}$ and then we obtain

$$y^{(2)}=\frac{\partial L_f h}{\partial x}\big(f(x)+g(x)u\big)=L_f^2 h(x)+L_g L_f h(x)u,$$

where

$$L_f^2 h(x)=\frac{\partial L_f h}{\partial x_1}f_1+\cdots+\frac{\partial L_f h}{\partial x_n}f_n,$$

$$L_g L_f h(x)=\frac{\partial L_f h}{\partial x_1}g_1+\cdots+\frac{\partial L_f h}{\partial x_n}g_n,$$

$$\begin{aligned}
\frac{\partial L_f h}{\partial x_1}=&\sum_{i_3=\sigma_3}^{\sigma_3+1}\omega_3^{i_3}\cdots\sum_{i_n=\sigma_n}^{\sigma_n+1}\omega_n^{i_n}\frac{h(\delta_1,\delta_2,\ldots,i_n)}{d_1(\delta_1)d_2(\delta_2)}f_2+\cdots\\
&+\sum_{i_2=\sigma_2}^{\sigma_2+1}\omega_2^{i_2}\cdots\sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1}\omega_{n-1}^{i_{n-1}}\frac{h(\delta_1,i_2,\ldots,\delta_n)}{d_1(\delta_1)d_n(\delta_n)}f_n\\
&+\sum_{i_2=\sigma_2}^{\sigma_2+1}\omega_2^{i_2}\cdots\sum_{i_n=\sigma_n}^{\sigma_n+1}\omega_n^{i_n}\frac{h(\delta_1,i_2,\ldots,i_n)}{d_1(\delta_1)}\\
&\times\sum_{i_2=\sigma_2}^{\sigma_2+1}\omega_2^{i_2}\cdots\sum_{i_n=\sigma_n}^{\sigma_n+1}\omega_n^{i_n}\frac{f_1(\delta_1,i_2,\ldots,i_n)}{d_1(\delta_1)}+\cdots\\
&+\sum_{i_1=\sigma_1}^{\sigma_1+1}\omega_1^{i_1}\cdots\sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1}\omega_{n-1}^{i_{n-1}}\frac{h(i_1,i_2,\ldots,\delta_n)}{d_n(\delta_n)}\\
&\times\sum_{i_2=\sigma_2}^{\sigma_2+1}\omega_2^{i_2}\cdots\sum_{i_n=\sigma_n}^{\sigma_n+1}\omega_n^{i_n}\frac{f_n(\delta_1,i_2,\ldots,i_n)}{d_1(\delta_1)}
\end{aligned}\tag{7}$$

$$\begin{aligned}
\frac{\partial L_f h}{\partial x_n}=&\sum_{i_2=\sigma_2}^{\sigma_2+1}\omega_2^{i_2}\cdots\sum_{i_n=\sigma_{n-1}}^{\sigma_{n-1}+1}\omega_{n-1}^{i_{n-1}}\frac{h(\delta_1,i_2,\ldots,\delta_n)}{d_1(\delta_1)d_n(\delta_n)}f_1\\
&+\cdots+\sum_{i_1=\sigma_1}^{\sigma_1+1}\omega_1^{i_1}\cdots\sum_{i_{n-2}=\sigma_{n-2}}^{\sigma_{n-2}+1}\omega_{n-2}^{i_{n-2}}\frac{h(i_1,\ldots,\delta_{n-1},\delta_n)}{d_{n-1}(\delta_{n-1})d_n(\delta_n)}f_{n-1}\\
&+\sum_{i_2=\sigma_2}^{\sigma_2+1}\omega_2^{i_2}\cdots\sum_{i_n=\sigma_n}^{\sigma_n+1}\omega_n^{i_n}\frac{h(\delta_1,i_2,\ldots,i_n)}{d_1(\delta_1)}\\
&\times\sum_{i_1=\sigma_1}^{\sigma_1+1}\omega_1^{i_1}\cdots\sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1}\omega_{n-1}^{i_{n-1}}\frac{f_1(i_1,i_2,\ldots,\delta_n)}{d_n(\delta_n)}+\cdots\\
&+\sum_{i_1=\sigma_1}^{\sigma_1+1}\omega_1^{i_1}\cdots\sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1}\omega_{n-1}^{i_{n-1}}\frac{h(i_1,i_2,\ldots,\delta_n)}{d_n(\delta_n)}\\
&\times\sum_{i_1=\sigma_1}^{\sigma_1+1}\omega_1^{i_1}\cdots\sum_{i_{n-1}=\sigma_{n-1}}^{\sigma_{n-1}+1}\omega_{n-1}^{i_{n-1}}\frac{f_n(i_1,i_2,\ldots,\delta_n)}{d_n(\delta_n)}
\end{aligned}\tag{8}$$

Once again, if $L_g L_f h(x)=0$, then $y^{(2)}=L_f^2 h(x)$ is independent of u. Repeating this process, we see that if $h(x)$ satisfies

$$L_g L_f^i h(x)=0,\qquad i=0,1,\ldots,\rho-2,$$
$$L_g L_f^{\rho-1}h(x)\neq 0$$

then u does not appear in the equations of $\dot{y}$, $\ddot{y}$, ... , $y^{(\rho-1)}$ and appears in the equation of $y^{(\rho)}$ with a nonzero coefficient:

$$y^{(\rho)}=L_f^\rho h(x)+L_g L_f^{\rho-1}h(x)u.$$

The foregoing equation shows clearly that the system is input-output linearizable, since the state feedback control

$$u=(-L_f^\rho h(x)+v)/L_g L_f^{\rho-1}h(x)$$

reduces the input-output map to $y^{(\rho)}=v$, which is a chain of ρ integrators. In this case, the integer ρ is called the relative degree of the system.

If $L_g L_f^{\rho-1}h(x_t)=0$, the relative degree cannot be defined at $x=x_t$. In some cases the relative degree can be defined at the point because we can adjust a partition of the state space for PB modeling so that $L_g L_f^{\rho-1}h(x_t)\neq 0$.

Definition 3.1: The nonlinear system is said to have relative degree ρ, $1\le\rho\le n$, in a region $D_0\subset D$ if

$$\begin{aligned}&L_g L_f^i h(x)=0,\ i=0,1,\cdots,\rho-2\\&L_g L_f^{\rho-1}h(x)\neq 0,\end{aligned}\tag{9}$$

for all $x\in D_0$..

The input-output linearized system can be formulated as

$$\begin{cases}\dot{\xi}=A\xi+Bv,\\ y=C\xi,\end{cases}\tag{10}$$

where $\xi\in\Re^\rho$, $C=(1,0,\ldots,0,0)^T$,

$$A=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\ddots&\vdots\\\vdots&\vdots&\ddots&\ddots&0\\0&0&\cdots&0&1\\0&0&\cdots&0&0\end{pmatrix},\quad B=\begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}.\tag{11}$$

Note that all the PB models (5) are transformed into the linear system (10). Therefore it is easy to design the stabilizing controller and analyze the stability of the PB systems.

According to the relative degree, three cases of linearized systems (10) must be considered.

Relative degree: $\rho=n$

In this case, the state vector of the input-output linearized system is

$z = \xi = (h(x), L_f h(x), \cdots, L_f^{\rho-1} h(x))^T$

The state vector z is necessary to be a diffeomorphism.

Relative degree: $\rho < n$

There is unobservable state (n - ρ dimensions). It is necessary to consider the zero dynamics of the unobservable state μ. The state vector z is necessary to be a diffeomorphism.

$$z = (\xi, \mu)^T, \quad \xi \in \Re^{\rho}, \mu \in \Re^{n-\rho},$$
$$\dot{\mu}(\xi, \mu) = \zeta_1(\xi, \mu) + \zeta_2(\xi, \mu)v.$$

$\dot{\mu}(0, \mu)$ is characterized by zero dynamics.

In the case of $L_g L_f^i h(x) = 0$, $\forall i$, the proposed approach cannot be applied.

When the relative degree $\rho \le n$, the input-output linearizing controller is $u = \alpha(x) + \beta(x)v$, where

$$\alpha(x) = -L_f^{\rho} h(x) / L_g L_f^{\rho-1} h(x), \ \beta(x) = 1 / L_g L_f^{\rho-1} h(x).\,.$$

In the following, we assume the relative degree is n (full). The stabilizing linear controller $v = -F\xi$ of the linearized system (10) can be obtained so that the transfer function $G = C(sI - A)^{-1} B$ is Hurwitz.

The linearizing controller can be characterized as the LUT (Look-Up-Table) controller, where the LUT-controller is widely used for industrial applications, in particular, for vehicle control because of simplicity and also visibility as a nonlinear controller. In the case of the LUT-controller, control inputs are calculated by interpolation based on the table. When bilinear piecewise interpolation is adopted, the LUT-controller is found to be exactly the PB system.

IV. Observer-Based PB Controller

We propose an observer-based PB controller to estimate the coordinate transformation z by using the error of $y - \hat{y}$. In this paper, we construct a full-order state observer for PB control system as shown in Fig. 2. In this figure, T and $1/S$ show the coordinate transformation and the integrator. F is the feedback gain, K is the observer gains, and r ($\dot{r} = 0$) is the set point signal. Due to lack of space, we only discuss the nonlinear system with the relative degree $\rho = n$. The following approach can be also applied to the nonlinear systems with $\rho < n$. We consider the linearized system using PB models.

$$\begin{cases} \dot{z} = Az + Bv, \\ y = Cz \end{cases} \tag{12}$$

The full-order state observer system is of a following form.

$$\begin{cases} \dot{\hat{z}} = A\hat{z} + Bv + K(y - \hat{y}), \\ \hat{y} = C\hat{z}, \end{cases} \tag{13}$$

where the controller

$$v = -F\hat{z}. \tag{14}$$

The closed loop system including the systems (12) and (13) is obtained as

$$\begin{pmatrix} \dot{z} \\ \dot{\hat{z}} \end{pmatrix} = \begin{pmatrix} A & -BF \\ KC & A - BF - KC \end{pmatrix} \begin{pmatrix} z \\ \hat{z} \end{pmatrix}$$

The following closed loop system can be obtained by using a coordinate transformation $e = \hat{z} - z$ and the similarity transformation.

$$\begin{pmatrix} \dot{z} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A - BF & -BF \\ 0 & A - KC \end{pmatrix} \begin{pmatrix} z \\ e \end{pmatrix}$$

We can design the feedback gain F and observer gain K separately. This is based on the separation principle. In general, one selects the poles of $A - KC$ on the left side of the poles of $A - BF$. Finally, the observer-based PB controller is obtain as

$$u = \alpha(x) + \beta(x)v = \frac{-L_f^{\rho} h(x) - F\hat{z}}{L_g L_f^{\rho-1} h(x)}$$

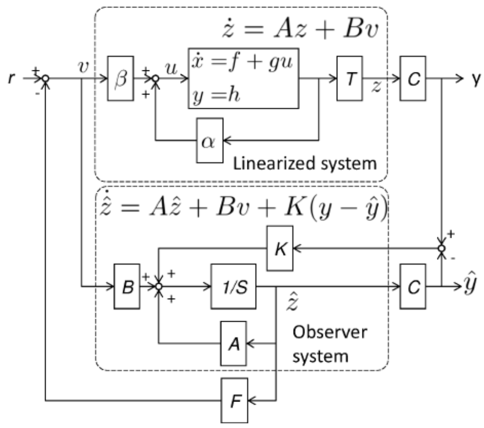

Fig. 2. Full-order state observer of PB control system.

V. Numerical Examples

Consider the nonlinear system

$$\begin{cases} \dot{x} = f_o(x) + g_o u(x) = \begin{pmatrix} -x_1 + x_2 + \frac{1}{2} x_2^2 \\ -x_3 + \frac{1}{3} x_3^3 \\ x_1^2 - x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \\ y = h_o(x) = x_1. \end{cases} \tag{15}$$

The conventional I/O feedback linearization controller is designed as

$$u(x) = \frac{-L_{f_o}^3 h_o(x)}{L_{g_o} L_{f_o}^2 h_o(x)} + \frac{1}{L_{g_o} L_{f_o}^2 h_o(x)} v,$$

where

$$L_{f_o}^3 h_o(x) = f_{o_1}(x) - (1+x_2) f_{o_2}(x) + (1+x_2)(-1+x_3^2) f_{o_3}(x),$$

$$L_{g_o} L_{f_o}^2 h_o(x) = (1+x_2)(-1+x_3^2),$$

$$f_o(x) = \left(f_{o_1}(x), f_{o_2}(x), f_{o_3}(x) \right)^T.$$

The linearized controller cannot be applied to stabilize the nonlinear system (15) at $x_2 = -1$ and $x_3 = \pm 1$ since $L_{g_o} L_{f_o}^2 h_o(x) = 0$ at these points. Therefore, the linearizable region is restricted to $x_2 \in [-1, \infty)$ and $x_3 \in [-1, 1]$. Divide the state space of the nonlinear system (15) as $x_1 = \{-3, -1.5, \cdots, 3\}$, $x_2 = \{-2, -1.2, 0, 1.2, 2\}$ and $x_3 = \{-2, -1.2, 0, 1.2, 2\}$, then the PB model is constructed as

$$\begin{cases} \dot{x} = f(x) + gu(x) = \left(f_1(x), f_2(x), f_3(x) \right)^T + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \\ y = h(x) = x_1. \end{cases}$$

where

$$f_1(x) = \sum_{i=\sigma}^{\sigma+1} \sum_{j=\tau}^{\tau+1} \omega_1^i(x_1) \omega_2^j(x_2) f_1(i, j, \cdot),$$

$$f_2(x) = \sum_{k=\upsilon}^{\upsilon+1} \omega_3^k(x_3) f_2(\cdot, \cdot, k),$$

$$f_3(x) = \sum^{\sigma+1} \sum^{\upsilon+1} \omega_1^i(x_1) \omega_3^k(x_3) f_3(i, \cdot, k).$$

Here, $f_1(i, j, \cdot)$ means that it is independent of k and also about $f_2(\cdot, \cdot, k)$ and $f_3(i, \cdot, k)$.

TABLE I: PB MODELS

$f_1(i,j,\cdot)$	$d_1(-2)$	$d_1(-1)$	$d_1(0)$	$d_1(1)$	$d_1(2)$
$d_2(-2)$	3.00	1.50	0	-1.50	-3.00
$d_2(-1)$	2.52	1.02	-0.48	-1.98	-3.48
$d_2(0)$	3.00	1.50	0	-1.50	-3.00
$d_2(1)$	4.92	3.42	1.92	0.42	-1.08
$d_2(2)$	7.00	5.50	4.00	2.50	1.00
$f_2(i,\cdot,k)$	$d_1(-2)$	$d_1(-1)$	$d_1(0)$	$d_1(1)$	$d_1(2)$
$d_3(-2)$	-0.667	-0.667	-0.667	-0.667	-0.667
$d_3(-1)$	0.624	0.624	0.624	0.624	0.624
$d_3(0)$	0	0	0	0	0
$d_3(1)$	-0.624	-0.624	-0.624	-0.624	-0.624
$d_3(2)$	0.667	0.667	0.667	0.667	0.667
$f_3(i,\cdot,k)$	$d_1(-2)$	$d_1(-1)$	$d_1(0)$	$d_1(1)$	$d_1(2)$
$d_3(-2)$	11.0	4.25	2.00	4.25	11.0
$d_3(-1)$	10.2	3.45	1.20	3.45	10.2
$d_3(0)$	9.00	2.25	0	2.25	9.00
$d_3(1)$	7.80	1.05	-1.20	1.05	7.80
$d_3(2)$	7.00	0.25	-2.00	0.25	7.00

Table I shows the PB models of $f_1(x)$, $f_2(x)$ and $f_3(x)$. We design the stabilizing controller:

$$u(x) = \frac{-L_f^3 h(x)}{L_g L_f^2 h(x)} + \frac{1}{L_g L_f^2 h(x)} v, \tag{16}$$

where

$$v = -Fz = -\left(2.12, \quad 5.27, \quad 2.79\right) z,$$

$$z = \left(x_1, f_1(x), \frac{\partial f_1(x)}{\partial x_1} f_1(x) + \frac{\partial f_1(x)}{\partial x_2} f_2(x) \right)^T,$$

$$L_f^3 h(x) = \left(\frac{\partial f_1(x)}{\partial x_1} \right)^2 f_1(x) + \frac{\partial f_1(x)}{\partial x_1} \frac{\partial f_1(x)}{\partial x_2} f_2(x) + \frac{\partial f_1(x)}{\partial x_2} \frac{\partial f_2(x)}{\partial x_3} f_3(x),$$

$$L_g L_f^2 h(x) = \frac{\partial f_1(x)}{\partial x_2} \frac{\partial f_2(x)}{\partial x_3},$$

$$\frac{\partial f_1(x)}{\partial x_1} = (f_1(\sigma+1, \cdot, \cdot) - f_1(\sigma, \cdot, \cdot))/d_1(\Delta_\sigma),$$

$$\frac{\partial f_1(x)}{\partial x_2} = (f_1(\cdot, \tau+1, \cdot) - f_1(\cdot, \tau, \cdot))/d_2(\Delta_\tau),$$

$$\frac{\partial f_2(x)}{\partial x_3} = (f_2(\cdot, \cdot, \upsilon+1) - f_2(\cdot, \cdot, \upsilon))/d_3(\Delta_\upsilon).$$

Note that $\partial f_1(x_1)/\partial x_1$, $\partial f_1(x_1)/\partial x_2$ and $\partial f_2(x_1)/\partial x_2$ in the stabilizing controller (16) are constant parameters. The stabilizing controller is represented as a bilinear function. Therefore the controller is characterized as the LUT controller.

From the original $f_1(x)$ and $f_2(x)$, we can derive

$$\begin{aligned} &f_1(\cdot, \tau+1, \cdot) - f_1(\cdot, \tau, \cdot) \\ =&[1 + 1/2(d_2(\tau) + d_2(\tau+1))](d_2(\tau+1) - d_2(\tau)), \\ &f_2(\cdot, \cdot, \upsilon+1) - f_2(\cdot, \cdot, \upsilon) \\ =&[-1 + 1/3(d_3(\upsilon)^2 + d_3(\upsilon) d_3(\upsilon+1) + d_3(\upsilon+1)^2)] \\ &\times (d_3(\upsilon+1) - d_3(\upsilon)). \end{aligned}$$

If $1 + 1/2(d_2(\tau) + d_2(\tau+1)) \neq 0$ and $-1 + 1/3(d_3(\upsilon)^2 + d_3(\upsilon) d_3(\upsilon+1) + d_3(\upsilon+1)^2) \neq 0$, then $L_g L_f^2 h(x) \neq 0$. These conditions are satisfied by the current partition of the state space. Therefore, the PB model is found to be globally feedback linearizable. Fig. 3 shows a simulation result of the initial condition $x(0) = (1, 1.5, 1)^T$, where it is seen that x_3 exceeds 1: note that the original linearization region is $-1 < x_3 < 1$.

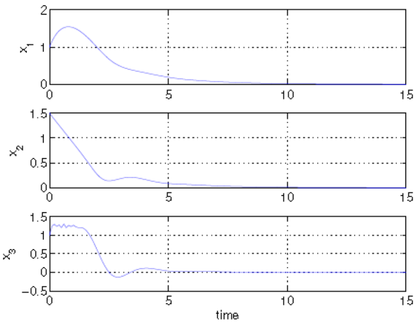

Fig. 3. Control responses using the PB controller.

We apply the observer design method to this nonlinear system. We select the poles of $A - KC$ on the left side of the poles of $A - BF$, then the feedback gain is calculated as $F = (2.12, 5.27, 2.79)$ and observer gain is calculated as $K = (33.0, 362, 1320)^T$. Fig. 4 shows a simulation result of the initial condition $x(0) = (1, 1.5, 1)^T$, where it is also seen that x_3 exceeds 1: note that the original linearization region is $-1 < x_3 < 1$. Fig. 5 shows the estimated state trajectories of PB control with the state observer.

Note that the observer-based PB controller is simpler than the conventional feedback linearization one. Since the nonlinear terms of the controller (16) are not the original nonlinear terms (e.g., x_1^2, x_2^2 and x_3^3) and the PB controller is represented as an LUT one. However the control performance based on PB model is the same as the conventional one.

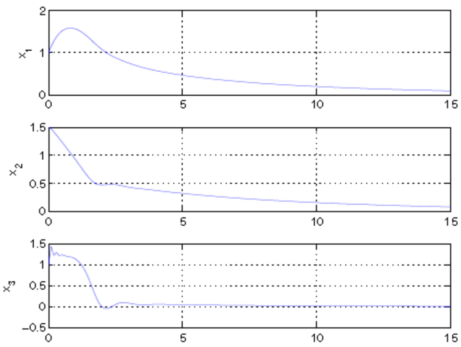

Fig. 4. State responses using the observer-based PB controller.

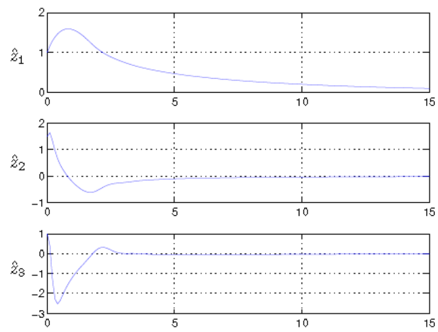

Fig. 5. Estimated state trajectories using the observer-based PB controller.

VI. Conclusion

This paper has proposed a full-order state observer design of nonlinear control systems approximated by piecewise bilinear (PB) models. We have constructed PB model of nonlinear control systems. The approximated system is found to be fully parametric. The input-output (I/O) feedback linearization has been applied to stabilize PB control systems. The design method is capable of designing the state observer and the servo controller of nonlinear systems separately. Although the PB controller is simpler than the conventional I/O feedback linearization controller, the control performance based on PB model is the same as the conventional one. We have presented the PB modeling combined with the conventional feedback linearization as a very powerful tool for the analysis and synthesis of nonlinear control systems. An illustrative example has confirmed the feasibility of our proposals.

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