Iterates Properties for q-Bernstein-Stancu Operators

Yali Wang and Yinying Zhou

***Abstract*—In 2009, Nowak introduced q-Bernstein-Stancu polynomials $B_n^{q,\alpha}(f;x)$. When $\alpha=0$, $B_n^{q,\alpha}(f;x)$ reduces to the well-known q-Bernstein polynomials introduced by Phillips in 1997; when $q=1$, $B_n^{q,\alpha}(f;x)$ reduces to Bernstein-Stancu polynomials introduced by Stancu in1968; when $q=1$ and $\alpha=0$, we obtain classical Bernstein polynomials. This paper deals with iterates properties of q-Bernstein-Stancu operators $B_n^{j_n}(f,q;x)$ in the case $q\in(0,1)$, $\alpha>0$, $f\in\mathbb{C}[0,1]$, where both $j_n\to\infty$ and $n\to\infty$.**

***Index Terms*—q-Bernstein-Stancu polynomials, iterates properties, uniform convergence.**

I. Introduction

Let $q>0$, for each nonnegative integer r, we define the $q-$integer $[r]$ as

$$[r]=[r]_q:=\begin{cases}\dfrac{(1-q^r)}{(1-q)}, & q\neq 1,\\ r, & q=1.\end{cases}$$

We then define $q-$factorial $[r]!$ as

$$[r]!=[r]_q!:=[r][r-1]\ldots[1],[0]!=1.$$

We next define a $q-$binomial coefficient as

$$\begin{bmatrix}n\\r\end{bmatrix}=\begin{bmatrix}n\\r\end{bmatrix}_q:=\frac{[n][n-1]\ldots[n-r-1]}{[r]!}=\frac{[n]!}{[r]![n-r]!}$$

For $f\in\mathbb{C}[0,1]$, $q>0,\alpha\geq 0$ and each positive integer n, we shall investigate the following q-Bernstein-Stancu operator introduced by Nowak in 2009 [1].

$$B_n^{q,\alpha}(f;x)=\sum_{k=0}^{n}B_{n,k}^{q,\alpha}(x)f\left(\frac{[k]}{[n]}\right) \tag{1}$$

where

$$B_{n,k}^{q,\alpha}(x)=\begin{bmatrix}n\\k\end{bmatrix}\frac{\prod_{i=0}^{k-1}(x+\alpha[i])\prod_{j=0}^{n-k-1}(1-q^jx+\alpha[j])}{\prod_{j=0}^{n-1}(1+\alpha[j])}, \tag{2}$$

Note that empty product in (2) denotes 1.

In this case, when $\alpha=0, B_n^{q,\alpha}(f;x)$ reduces to the well-known q-Bernstein polynomials introduced by Phillips [2] in1997:

$$B_{n,q}(f;x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^k\prod_{i=0}^{n-k-1}(1-q^ix)f\left(\frac{[k]}{[n]}\right).$$

when $q=1$, $B_n^{q,\alpha}(f;x)$ reduces to Bernstein-Stancu polynomials introduced by Stancu [3] in 1968:

$$S_n(f;x)=\sum_{k=0}^{n}\binom{n}{r}\frac{\prod_{i=0}^{k1}(x+\alpha i)\prod_{s=0}^{n-k-1}(1-x+s\alpha)}{\prod_{i=0}^{n-1}(1+i\alpha)}f\left(\frac{k}{n}\right)$$

when $q=1$, $\alpha=0$, we obtain the classical Bernstein polynomials defined by

$$B_n(f;x)=\sum_{k=0}^{n}\binom{n}{r}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right)$$

Now, we review and state some general properties of q-Bernstein-Stancu operators.

It follows directly from the definition that q-Bernstein-Stancu operators possess the end-point interpolation property, i.e.,

$$B_n^{q,\alpha}(f;0)=f(0), B_n^{q,\alpha}(f;1)=f(1), \qquad q>0, \qquad n\in \tag{3}$$

and leave invariant linear function, that is

$$B_n^{q,\alpha}(at+b)=ax+b \qquad q>0, \qquad n\in \tag{4}$$

They are also degree-reducing on polynomials, that is if $\mathcal{P}_m$ is a polynomial of degree m, then $B_n^{q,\alpha}(\mathcal{P}_m)$ is a polynomial of degree $\leq\min(m,n)$.

Taking $a=0$, $b=1$ in (4), we conclude that

$$\sum_{k=0}^{n}B_{n,k}^{q,\alpha}(x)=1, \qquad n\in\mathbb{N} \tag{5}$$

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In 2009, Nowak proved that the q-Bernstein-Stancu operators can be expressed in terms of q-differences [1]:

$$B_n^{q,\alpha}(f;x)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\Delta_q^k f_0\prod_{s=0}^{k-1}\frac{x+\alpha[s]}{1+\alpha[s]}, \tag{6}$$

where

$$\Delta_q^k f_0=\frac{[k]!}{[n]^k}q^{\frac{k(k-1)}{2}}f\left[0;\frac{1}{[n]};\ldots,\frac{[k]}{[n]}\right].$$

Then

$$B_n^{q,\alpha}(f;x)=\sum_{k=0}^{n}\lambda_{k,q}^{n}f\left[0;\frac{1}{[n]},\ldots,\frac{[k]}{[n]}\right]\prod_{i=0}^{k-1}\frac{x+\alpha[i]}{1+\alpha[i]}, \tag{7}$$

where

$$\lambda_{k,q}^{(n)}:=\begin{bmatrix} n \\ k \end{bmatrix}\frac{[k]!}{[n]^q}q^{\frac{k(k-1)}{2}}=\left(1-\frac{1}{[n]}\right)\ldots\left(1-\frac{[k-1]}{[n]}\right). \tag{8}$$

Note that

$$\lambda_{0,q}^{(n)}=\lambda_{1,q}^{(n)}=1, \tag{9}$$

and

$$0\le\lambda_{k,1}\le 1,\qquad k=0,1,\ldots,n. \tag{10}$$

at the same time, he still prove that for $0<q<1,\ \alpha>0,$

$$B_n^{q,\alpha}(1;x)=1,\ B_n^{q,\alpha}(t;x)=x, \tag{11}$$

and

$$B_n^{q,\alpha}(t^2;x)=\frac{1}{1+\alpha}\left(x(x+\alpha)+\frac{x(1-x)}{[n]}\right). \tag{12}$$

He also proved some other approximation properties [1].

In recent years, q-Bernstein polynomials have been studied intensively by a number of authors. They investigated iterates properties of the Bernstein operator from a different point of view [4]–[6].

We will deal with iterates properties of q-Bernstein-Stancu operators $B_n^{j_n}(f,q;x)$ in the case $q\in(0,1),\ \ \alpha>0,\ f\in\mathbb{C}[0,1]$, where both $j_n\to\infty$ and $n\to\infty$ in this paper.

It can be readily seen that for $q\in(0,1)$, polynomials $B_{n,k}^{q,\alpha}$ are non negative on the interval $[0,1]$. Therefore, we get from $\sum_{k=0}^{n}B_{n,k}^{q,\alpha}(x)=1$ that

$$\left\|B_n^{q,\alpha}\right\|=1,\qquad q\in(0,1). \tag{13}$$

In this paper we always assume that $B_\infty^{q,\alpha}$ on $\mathbb{C}[0,1]$ exists.

We denote the operator of linear interpolation at 0 and 1 by L, i.e.,

$$L(f;x):=(1-x)f(0)+xf(1)\cdot$$

Now we give the statement of main results in this paper.

Theorem1. For $q\in(0,1),\ \alpha>0$, let $\{j_n\}$ be a sequence of positive integers such that $j_n\to\infty$. Then for any $f\in\mathbb{C}[0,1]$,

$$\left(B_{n,q}^{\alpha}\right)^{j_n}\rightrightarrows L(f;x)\ \text{ for } x\in[0,1]\ \text{ as } n\to\infty.$$

II. Proof of Theorem 1

Lemma 1. Let $f=t^m,\ m\ge 1,$ then

$$\left(B_n^{q,\alpha}\right)(f;x)$$
$$=\alpha_1\prod_{i=1}^{0}\frac{x+\alpha[i]}{1+\alpha[i]}+\alpha_2\prod_{i=0}^{1}\frac{x+\alpha[i]}{1+\alpha[i]}+\ldots+\alpha_j\prod_{i=0}^{j-1}\frac{x+\alpha[i]}{1+\alpha[i]},\ j=\min(m,n), \tag{14}$$

where

all $\alpha_i\ge 0\,(i=1,\ldots,j).$

$$\alpha_1+\alpha_2+\ldots+\alpha_j=1.$$

Besides, for $n\ge m$, we have

$$\alpha_i\le\frac{C_{i,m}}{[n]_q^{m-i}},\qquad i=1,\ldots,m.$$

$$\alpha_m=\lambda_{m,q}^{(n)},\quad \alpha_{m-1}=\lambda_{m-1,q}^{(n)}\frac{1+[2]+\ldots+[m-1]}{[n]}.$$

Proof. It was already noticed in the introduction $B_n^{q,\alpha}(t^m;x)$ is a polynomials of degree min (m,n). The end-point interpolation property (3) implies that for $m\ge 1$, the free term of $B_n^{q,\alpha}(t^m;x)$ equals 0. Therefore, (14) is justified.

1) Representation (7) of q-Bernstein-Stancu polynomials gives the following values of coefficients in (14)

$$\alpha_i=\lambda_{i,q}^{(n)}f\left[0,\frac{1}{[n]},\ldots,\frac{[i]}{[n]}\right],\qquad i=1,\ldots,m \tag{15}$$

where $0\le\lambda_{i,q}^{(n)}\le 1$ are given by (10) Since for $f=t^m,\ f\left[0,\frac{1}{[n]},\ldots,\frac{[i]}{[n]}\right]\ge 0,$ the statement is proved.

2) This follows readily from the end-point interpolation properties (3) if we put $x=1$ in (14).

3) Using (15) and (10) we get

$$\alpha_i \le f\left[0,\frac{1}{[n]},\ldots,\frac{[i]}{[n]}\right]=\frac{f^{(i)}(\xi_i)}{i!},\ \text{where}\quad \xi_i\in\left(0,\frac{[i]}{[n]}\right)$$

Hence

$$\alpha_i \le \binom{m}{i}\xi_i^{m-1}\le\binom{m}{i}\left(\frac{[i]}{[n]}\right)^{m-i}=:\frac{c_{m,i}}{[n]_q^{m-i}},$$

as required.

4) The proof see [7].

Lemma 2. For all $q\in(0,1)$, $\alpha>0$, the following identity holds:

$$B_n^{q,\alpha}\left(t^m;x\right)=\frac{x}{[n]^{m-1}\left(1+\alpha[n-1]\right)}\sum_{j=0}^{m-1}\begin{bmatrix}m-1\\ j\end{bmatrix}\left(q[n-1]\right)^j B_{n-1}^{q,\alpha}\left(t^i;x\right)+$$

$$\frac{\alpha}{[n]^{m-1}\left(1+\alpha[n-1]\right)}\sum_{j=0}^{m-1}\begin{bmatrix}m-1\\ j\end{bmatrix}q^j[n-1]^{j+1}B_{n-1}^{q,\alpha}\left(t^{j+1};x\right).\qquad(16)$$

Proof. Let $B_{n,k}^{q,\alpha}(x)$ be defined by (2) Then

$$B_n^{q,\alpha}\left(t^m;x\right)=\sum_{k=0}^{n}\left(\frac{[k]}{[n]}\right)^m B_{n,k}^{q,\alpha}(x)$$

$$=\sum_{k=1}^{n}\left(\frac{[k]}{[n]}\right)^{m-1}\left(\frac{[k]}{[n]}\right)\begin{bmatrix}n\\ k\end{bmatrix}\frac{\prod_{i=0}^{k-1}\left(x+\alpha[i]\right)\prod_{s=0}^{n-k-1}\left(1-q^s x+\alpha[s]\right)}{\prod_{i=0}^{n-1}\left(1+\alpha[i]\right)}$$

$$=\sum_{k=1}^{n}\left(\frac{[k]}{[n]}\right)^{m-1}\begin{bmatrix}n-1\\ k-1\end{bmatrix}\frac{\prod_{i=0}^{k-1}\left(x+\alpha[i]\right)\prod_{s=0}^{n-k-1}\left(1-q^s x+\alpha[s]\right)}{\prod_{i=0}^{n-1}\left(1+\alpha[i]\right)}$$

$$=\sum_{k=0}^{n-1}\left(\frac{[k+1]}{[n]}\right)^{m-1}\begin{bmatrix}n-1\\ k\end{bmatrix}\frac{\prod_{i=0}^{(k+1)-1}\left(x+\alpha[i]\right)\prod_{s=0}^{n-(k+1)-1}\left(1-q^s x+\alpha[s]\right)}{\prod_{i=0}^{(n-1)-1}\left(1+\alpha[i]\right)\left(1+\alpha[n-1]\right)}$$

$$=\frac{\sum_{k=0}^{n-1}\left(\sum_{j=0}^{m-1}\begin{bmatrix}m-1\\ j\end{bmatrix}\left(\frac{q[k][n-1]}{[n-1]}\right)^j\right)\left(x+\alpha[k]\right)B_{n-1,k}^{q,\alpha}(x)}{[n]^{m-1}\left(1+\alpha[n-1]\right)}$$

$$\frac{\sum_{j=0}^{m-1}\begin{bmatrix}m-1\\ j\end{bmatrix}\left(q[n-1]\right)^j\sum_{k=0}^{n-1}\left(\frac{[k]}{[n-1]}\right)^j\left(x+\alpha[k]\right)B_{n-1,k}^{q,\alpha}(x)}{[n]^{m-1}\left(1+\alpha[n-1]\right)}$$

$$=\frac{\sum_{j=0}^{m-1}\begin{bmatrix}m-1\\ j\end{bmatrix}\left(q[n-1]\right)^j\sum_{k=0}^{n-1}\left(\frac{[k]}{[n-1]}\right)^j\left(x+\alpha[k]\right)B_{n-1,k}^{q,\alpha}(x)}{[n]^{m-1}\left(1+\alpha[n-1]\right)}$$

$$=\frac{x}{[n]^{m-1}\left(1+\alpha[n-1]\right)}\sum_{j=0}^{m-1}\begin{bmatrix}m-1\\ j\end{bmatrix}\left(q[n-1]\right)^j B_{n-1}^{q,\alpha}\left(t^j;x\right)+$$

$$\frac{\alpha}{[n]^{m-1}\left(1+\alpha[n-1]\right)}\sum_{j=0}^{m-1}\begin{bmatrix}m-1\\ j\end{bmatrix}q^j[n-1]^{j+1}B_{n-1}^{q,\alpha}\left(t^j;x\right).$$

Lemma 3. For all $q\in(0,1)$, $\alpha>0$, the operator $B_n^{q,\alpha}$ has $n+1$ linearly independent moni eigenvectors $P_m^{(n)}(x)$, deg $P_m^{(n)}(x)=m$, $(m=0,\ldots,n)$, corresponding to the eigenvalues

$$\lambda_{0,q}^{(n)}=\lambda_{1,q}^{(n)}=1,$$

$$\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)}=\frac{1}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)}\left(1-\frac{[1]}{[n]}\right)\left(1-\frac{[2]}{[n]}\right),\ldots,\left(1-\frac{[m-1]}{[n]}\right),$$

for

$$m=2,\ldots,n.\qquad(17)$$

Proof. For $m=0,1$, the statement is obvious due to

$$B_n^{q,\alpha}\left(at+b,q;x\right)=ax+b.$$

For $n\ge m\ge 2$, using lemma 1, we write

$$B_n^{q,\alpha}\left(t^m,q;x\right)=\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)}x^m+P_{m-1}^{(x)}(x).$$

where $P_{m-1}^{(n)}(x)\in\mathcal{P}_{m-1}$ and $\lambda_{m,q}^{(n)}$ are given by (8).

To find an eigenvector $P_m^{(n)}(x)\in\mathcal{P}_m$ of the operator $B_n^{q,\alpha}$, we write $P_m^{(n)}=x^m+a_{m-1}x^{m-1}+a_{m-1}x^{m-2}+\ldots+a_1x$ and solve a linear system in unknown $a_1,\ldots,a_{m-1}$:

$$B_{n,q}^{\alpha}\left(x^m+a_{m-1}x^{m-1}+\ldots+a_1x\right)=\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)}\left(x^m+a_{m-1}x^{m-1}+\ldots+a_1x\right)$$

By lemma 1 and by letting the coefficients of x^s in the left equals the coefficients of x^s in the right, $s=1,\ldots,m-1$, and arrange, we have:

$$B_n^{q,\alpha}\left(x^m\right)=v_{1,0}x+v_{2,0}x^2+\ldots+v_{m-1,0}x^{m-1}+\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)}x^m,$$

$$a_{m-1}B_n^{q,\alpha}\left(x^{m-1}\right)=a_{m-1}\left(v_{1,1}x+v_{2,1}x^2+\ldots+v_{m-2,1}x^{m-2}+\frac{\lambda_{m-1,q}^{(n)}x^{m-1}}{\prod_{i=0}^{m-2}\left(1+\alpha[i]\right)}\right),$$

$$a_{m-2}B_n^{q,\alpha}\left(x^{m-2}\right)=a_{m-2}\left(v_{1,2}x+v_{2,2}x^2+\ldots+v_{m-3,2}x^{m-3}+\frac{\lambda_{m-2,q}^{(n)}x^{m-2}}{\prod_{i=0}^{m-3}\left(1+\alpha[i]\right)}\right),$$

$$\vdots \qquad \vdots$$

$$a_1 B_n^{q,\alpha}(x) = a_1 \left(\frac{\lambda_{1,q}^{(n)}}{\prod_{i=0}^{0}(1+\alpha[i])} \right).$$

Then

$$\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])} a_{m-1} = \nu_{m-1,0} + \frac{\lambda_{m-1,q}^{(n)}}{\prod_{i=0}^{m-2}(1+\alpha[i])} a_{m-1},$$

$$\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])} a_{m-2} = \nu_{m-2,0} + \nu_{m-2,1} a_{m-1} + \frac{\lambda_{m-2,q}^{(n)}}{\prod_{i=0}^{m-3}(1+\alpha[i])} a_{m-2},$$

$$\vdots \qquad \vdots$$

$$\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])} a_1 = \nu_{1,0} + \nu_{1,1} a_{m-1} + \nu_{1,2} a_{m-2} + \ldots + \frac{\lambda_{1,q}^{(n)}}{\prod_{i=0}^{0}(1+\alpha[i])} a_1.$$

Then,

$$-\nu_{m-1,0} = \left(\frac{\lambda_{m-1,q}^{(n)}}{\prod_{i=0}^{m-2}(1+\alpha[i])} - \frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])} \right) a_{m-1},$$

$$-\nu_{m-2,0} = \nu_{m-2,1} a_{m-1} + \left(\frac{\lambda_{m-2,q}^{(n)}}{\prod_{i=0}^{m-3}(1+\alpha[i])} - \frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])} \right) a_{m-2},$$

$$\vdots \qquad \vdots$$

$$-\nu_{1,0} = \nu_{1,1} a_{m-1} + \nu_{1,2} a_{m-2} + \ldots + \left(\frac{\lambda_{1,q}^{(n)}}{\prod_{i=0}^{0}(1+\alpha[i])} - \frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])} \right) a_1.$$

We get a triangular system whose determine equals

$$\prod_{k=1}^{m-1} \left(\frac{\lambda_{m-k,q}^{(n)}}{\prod_{i=0}^{m-k-1}(1+\alpha[i])} - \frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])} \right) \neq 0.$$

Hence there exists a unique monic polynomial of degree $2 \le m \le n$ which is a eigenvector of $B_n^{q,\alpha}$ with the eigenvalue

$$\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])}.$$

Corollary 1. For $2 \le m \le n$, and all $q \in (0,1)$, $\alpha > 0$, the operator $\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])} I - B_n^{q,\alpha}$, where I is the identity operator, is invertible on $\mathcal{P}_{m-1}$.

Lemma 4. For $q \in (0,1)$, $\alpha > 0$, the following equality holds:

$$\lim_{n\to\infty} \frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])} = \frac{q^{\frac{m(m-1)}{2}}}{\prod_{i=0}^{m-1}(1+\alpha[i])}.$$

Proof. The statement follows from Formula (17)

$$\lim_{n\to\infty} \frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])} = \frac{\lim_{n\to\infty}\left(1-\frac{[1]}{[n]}\right)\left(1-\frac{[2]}{[n]}\right)\ldots\left(1-\frac{[m-1]}{[n]}\right)}{\prod_{i=0}^{m-1}(1+\alpha[i])}$$

$$= \frac{\lim_{n\to\infty} \frac{\frac{q(1-q^{n-1})}{1-q}}{[n]} \frac{\frac{q^2(1-q^{n-2})}{1-q}}{[n]} \ldots \frac{\frac{q^{m-1}(1-q^{n-m+1})}{1-q}}{[n]}}{\prod_{i=0}^{m-1}(1+\alpha[i])}$$

$$= \frac{1}{\prod_{i=0}^{m-1}(1+\alpha[i])} q^{\frac{m(m-1)}{2}}.$$

Lemma 5. Let $q \in (0,1), \alpha > 0$, then for every $m = 0,1\ldots$, the operator $B_\infty^{q,\alpha}$ has an eigenvector $P_m(x)$ which is a monic polynomial of degree m, corresponding to the eigenvalue

$$\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}(1+\alpha[i])} = (1-q)^{m-1} q^{\frac{(m-1)m}{2}} \prod_{i=1}^{m-1} \frac{1}{1-q+\alpha-\alpha q^i}.$$

Proof. Taking the limit in

$$B_n^{q,\alpha}(t^m;x) = \frac{x\sum_{j=0}^{m-1} \begin{bmatrix} m-1 \\ j \end{bmatrix} (q[n-1])^j B_{n-1,q}^{\alpha}(t^i q,;x)}{[n]^{m-1}(1+\alpha[n-1])}$$ in(16), as $n \to \infty$,

and we note that

$$\frac{q^j [n-1]^j}{[n]^{m-1}(1+\alpha[n-1])} \to \frac{q^j (1-q)^{m-j}}{1-q+\alpha},$$

We get

$$\frac{x\sum_{j=0}^{m-1}\begin{bmatrix} m-1 \\ j \end{bmatrix} q^j [n-1]^j B_{n-1,q}^{\alpha}\left(t^j,q;x\right)}{[n]^{m-1}\left(1+\alpha[n-1]\right)}$$

$$\to x\sum_{j=0}^{m-1}\begin{bmatrix} m-1 \\ j \end{bmatrix} \frac{q^j(1-q)^{m-l}}{1-q+\alpha} B_{\infty,q}^{\alpha}\left(t^j,q;x\right).$$

Similarly, taking the limit in

$$\frac{\alpha\sum_{j=0}^{m-1}\begin{bmatrix} m-1 \\ j \end{bmatrix} q^j [n-1]^{j+1} B_{n-1}^{q,\alpha}\left(t^{j+1};x\right)}{[n]^{m-1}\left(1+\alpha[n-1]\right)}, \text{ as } n\to\infty,$$

$$\because \frac{\alpha q^j [n-1]^{j+1}}{[n]^{m-1}\left(1+\alpha[n-1]\right)} q^j [n-1]^{j+1} \to \frac{aq^j(1-q)^{m-j-1}}{1-q+\alpha},$$

$$\therefore \frac{\alpha\sum_{j=0}^{m-1}\begin{bmatrix} m \\ j \end{bmatrix} q^j [n-1]^{j+1} B_{n-1,k}^{q,\alpha}\left(t^{j+1};x\right)}{[n]^{m-1}\left(1+\alpha[n-1]\right)}$$

$$\to \frac{\alpha\sum_{j=0}^{m-1}\begin{bmatrix} m-1 \\ j \end{bmatrix} q^j (1-q)^{m-j-1} B_{\infty}^{q,\alpha}\left(t^{j+1};x\right)}{1-q-\alpha}.$$

Taking the limit in (16), we have

$$B_{\infty}^{q,\alpha}\left(t^m;x\right) = x\sum_{j=0}^{m-1}\begin{bmatrix} m-1 \\ j \end{bmatrix} \frac{q^j(1-q)^{m-j}}{1-q+\alpha} B_{\infty}^{q,\alpha}\left(t^j;x\right)$$

$$+\alpha\sum_{j=0}^{m-1}\begin{bmatrix} m-1 \\ j \end{bmatrix} \frac{q^j(1-q)^{m-j-1}}{1-q+\alpha} B_{\infty}^{q,\alpha}\left(t^{j+1};x\right).$$

The coefficience $\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)}$ of x^m in $B_{\infty}^{q,\alpha}\left(t^m;x\right)$ equals

$$\frac{q^{m-1}(1-q)}{1-q+\alpha}\frac{\lambda_{m-1,q}^{(n)}}{\prod_{i=0}^{m-2}\left(1+\alpha[i]\right)} + \frac{\alpha q^{m-1}}{1-q+\alpha}\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)},$$

This means

$$\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)} = \frac{(1-q)q^{m-1}}{1-q+\alpha-\alpha q^{m-1}}\frac{\lambda_{m-1,q}^{(n)}}{\prod_{i=0}^{m-2}\left(1+\alpha[i]\right)}.$$

Recursively

$$\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)} = (1-q)^{m-1} q^{\frac{(m-1)m}{2}} \prod_{i=1}^{m-1}\frac{1}{1-q+\alpha-\alpha q^i}.$$

We have shown that

$$B_{\infty,q}^{\alpha}\left(t^m,q;x\right) = \frac{\lambda_{m,q}^{(n)}x^m}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)} + P_{m-1}^{(n)}(x), P_{m-1}^{(n)}(x)\in\mathcal{P}_{m-1}.$$

The statement now follows from considering the equation

$$B_{\infty,q}^{\alpha}\left(P_m(x)\right) = \frac{\lambda_{m,q}P_m(x)}{\prod_{i=0}^{m-2}\left(1+\alpha[i]\right)}, m=2,3,\ldots \quad .$$

Corollary 2. For $m\ge 2$, and all $q\in(0,1),\alpha>0$, the operator $\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)}I - B_{\infty}^{q,\alpha}$ is invertible on $\mathcal{P}_{m-1}$.

Proof of Theorem 1. Because of (3) it suffices to prove that $\left(B_n^{q,\alpha}\right)(f)\rightrightarrows ax+b$ for some a and b as $n\to\infty$.

(I) First we consider the case $f=x^m$.

We will use induction on m. For $m=0,1$, the statement is obvious due to (4). Assume that $\left(B_n^{q,\alpha}\left(x^t\right)\right)^{j_n}\rightrightarrows\varphi_t\in\mathcal{P}_1$ for $t=0,1,\ldots,m-1$, and let $\frac{\lambda_{m,q}^{(n)}}{\prod_{i=0}^{m-1}\left(1+\alpha[i]\right)} = \Lambda_{m,q}^{(n)}$. Consider

$$B_n^{q,\alpha}\left(x^m\right) = \Lambda_{m,q}^{(n)}x^m + P_{m-1}^{(n)}, \tag{18}$$

where $\Lambda_{m,q}^{(n)}$ is given by (17) and $p_{m-1}^{(n)}\in\mathcal{P}_{m-1}$. Then

$$\left(B_n^{q,\alpha}\left(x^t\right)\right)^{j_n} = \left(\Lambda_{m,q}^{(n)}\right)^{j_n}x^m +$$
$$\left[\left(\Lambda_{m,q}^{(n)}\right)^{j_n-1}I + \left(\lambda_{m,q}^{(n)}\right)^{j_n-2}B_n^{q,\alpha} + \ldots\left(B_n^{q,\alpha}\right)^{j_n-1}\right]\left(P_{m-1}^{(n)}\right),$$

where I denotes identity operator. It follows from Lemma 4 that

$$\left(\Lambda_{m,q}^{(n)}\right)^{j_n}\to 0 \text{ as } n\to\infty.$$

The expression in the brackets is a linear operator on the space. $\mathcal{P}_{m-1}$

Consider the sequence of polynomials in $\mathcal{P}_{m-1}$,

$$y_{m-1}^{(n)} := \left[\left(\Lambda_{m,q}^{(n)}\right)^{j_n-1}I + \left(\Lambda_{m,q}^{(n)}\right)^{j_n-2}B_n^{q,\alpha} + \ldots\left(B_n^{q,\alpha}\right)^{j_n-1}\right]\left(P_{m-1}^{(n)}\right)$$

Then,

$$\left(\Lambda_{m,q}^{(n)} I - B_n^{q,\alpha}\right) y_{m-1}^{(n)} := \left(\Lambda_{m,q}^{(n)}\right)^{j_n} P_{m-1}^{(n)} - \left(B_n^{q,\alpha}\right)^{j_n} P_{m-1}^{(n)}.$$

It follows from (13) and (18) that $\left\|P_{m-1}^{(n)}\right\| \leq 2.$ Since $\left(\Lambda_{m,q}^{(n)}\right)^{j_n} \to 0$ as $n \to \infty,$ we have

$$\left(\Lambda_{m,q}^{(n)}\right)^{j_n} P_{m-1}^{(n)} \rightrightarrows 0 \quad \text{as} \quad n \to \infty,$$

it can be readily seen from (18) and Lemma 4 that

$$P_{m-1}^{(n)}(x) \rightrightarrows B_\infty^{q,\alpha}(x^m) - \frac{q^{\frac{m(m-1)}{2}} x^m}{\prod_{i=0}^{m-1}(1+\alpha[i])} =: Q_{m-1}(x) \in \mathcal{P}_{m-1}, \quad n \to \infty,$$

i.e.

$$P_{m-1}^{(n)}(x) = Q_{m-1}(x) + \delta_n(x),$$

where $Q_{m-1} \in \mathcal{P}_{m-1}$, and $\delta_n(x) \rightrightarrows 0$ as $n \to \infty.$

Thus, $\left(B_n^{q,\alpha}\right)^{j_n} P_{m-1}^{(n)} = \left(B_n^{q,\alpha}\right)^{j_n} Q_{m-1} + \left(B_n^{q,\alpha}\right)^{j_n} (\delta_n),$

where $\left\|\left(B_n^{q,\alpha}\right)^{j_n} (\delta_n)\right\| \leq \|\delta_n\|$, because of (13) . This means that $\left(B_n^{q,\alpha}\right)^{j_n} (\delta_n) \rightrightarrows \infty.$ as $n \to \infty.$

By the induction assumption

$$\left(B_n^{q,\alpha}\right)^{j_n} Q_{m-1} \rightrightarrows cx + d \in \mathcal{P}_1 \quad \text{as} \quad n \to \infty.$$

Therefore, $\left(\Lambda_{m,q}^{(n)} I - B_n^{q,\alpha}\right) y_{m-1}^{(n)} \rightrightarrows cx + d$ as $n \to \infty.$

or

$$\left(\Lambda_{m,q}^{(n)} I - B_n^{q,\alpha}\right) y_{m-1}^{(n)} = cx + d + \omega_n(x),$$

where $\omega \rightrightarrows 0$ as $n \to \infty.$

By corollary 1, the operator $\Lambda_{m,q}^{(n)} I - B_n^{q,\alpha}$ are invertible on $\mathcal{P}_{m-1}$ for $n \geq m$ and

$$\lim_{n\to\infty}\left(\Lambda_{m,q}^{(n)} I - B_n^{q,\alpha}\right) = \frac{1}{\prod_{i=0}^{m-1}(1+\alpha[i])} q^{\frac{m(m-1)}{2}} I - B_\infty^{q,\beta} =: A_{\infty,q},$$

where by corollary 2, $A_{\infty,q}$ is also invertible on $\mathcal{P}_{m-1}$. Hence

$$\left(\Lambda_{m,q}^{(n)} I - B_n^{q,\alpha}\right)^{-1} \to A_{\infty,q}^{-1} \quad \text{as} \quad n \to \infty,$$

and it follows that

$$\left\|\left(\Lambda_{m,q}^{(n)} I - B_n^{q,\alpha}\right)^{-1}\right\| \leq M \quad \text{for some } M > 0.$$

Therefore,

$$y_{m-1}^{(n)} = \left(\Lambda_{m,q}^{(n)} I - B_n^{q,\alpha}\right)^{-1} (cx + d) + \left(\Lambda_{m,q}^{(n)} I - B_n^{q,\alpha}\right)^{-1} (\omega_n).$$

Since $\left\|\left(\Lambda_{m,q}^{(n)} I - B_n^{q,\alpha}\right)^{-1} (\omega_n)\right\| \leq M \|\omega_n\| \to 0$ as $n \to \infty$ and $\left(\Lambda_{m,q}^{(n)} I - B_n^{q,\alpha}\right)^{-1} \to A_{\infty,q}^{-1}$ as $n \to \infty$, we conclude that

$$y_{m-1}^{(n)} \rightrightarrows A_{\infty,q}^{-1}(cx + d) := ax + b \in \mathcal{P}_1.$$

Thus, $B_{n,q}^{jn}(x^m) \rightrightarrows ax + b.$

The induction is completed and it follows that for any polynomial $\mathcal{P}$,

$$\left(B_n^{q,\alpha}\right)^{jn} (p; x) \to L(p; x) \quad \text{for } x \in [0,1], \text{ as } n \to \infty.$$

(II) Let $f \in \mathbb{C}[0,1]$, and let $\varepsilon > 0$ be given. Then $f(x) = p(x) + \delta(x)$, where $p \in \mathcal{P}$, and $\|\delta(x)\| < \varepsilon.$ We have

$$\left(B_n^{q,\alpha}\right)^{j_n} (f) = \left(B_n^{q,\alpha}\right)^{j_n} (p) + \left(B_n^{q,\alpha}\right)^{j_n} (\delta).$$

Since $\left(B_n^{q,\alpha}\right)^{j_n} (p) \rightrightarrows L(p)$, there exists $n_0 \in \mathbb{N}$ such that $\left\|\left(B_n^{q,\alpha}\right)^{j_n} (p) - L(P)\right\| < \varepsilon$ for all $n > n_0$.

Obviously, $\|\delta\| < \varepsilon$, and finishlly we obtain

$$\left\|\left(B_n^{q,\alpha}\right)^{j_n} (f) - L(f)\right\| \leq \left\|\left(B_n^{q,\alpha}\right)^{j_n} (p) - L(p)\right\| + \left\|\left(B_n^{q,\alpha}\right)^{j_n} (\delta)\right\| + \|\delta\| < 3\varepsilon, \text{ for all } n > n_0.$$

Thus, $\left\|\left(B_n^{q,\alpha}\right)^{j_n} - L(f)\right\| \leq \left\|\left(B_n^{q,\alpha}\right)^{j_n} (p) - L(p)\right\| +$

$$\left\|\left(B_n^{q,\alpha}\right)^{j_n} (\delta)\right\| + \|\delta\| < 3\varepsilon, \quad \text{for all } n > n_0.$$

Thus, $\left(B_n^{q,\alpha}\right)^{j_n} (f; x) \rightrightarrows L(f; x)$, for $x \in [0,1]$ as $n \to \infty.$ □

III. Conclusion

In this paper, iterates properties for q-Bernstein-Stancu operators are studied, the result of iterates properties for q-Bernstein -Stancu operators is obtained. This study is just a small step in this area. To make further progress in this direction, one could try to study other properties for q-Bernstein-Stancu operators, such as shape-preserving and convergence properties to make this area perfected and enriched.

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References

[1] G. Nowak, "Approximation properties for generalized q-Bernstein polynomials," *Math. Anal*., pp. 1-10, Sep. 2009.

[2] G. M. Phillips, "Bernstein polynomials based on the q-integers," *Ann. Numer. Math.4*, pp. 511-518, 1997.

[3] D. D. Stancu, "Approximation of function by a new class of linear polynomials operators," *Rev. Roumaine Math. Pure Appl.* vol. XIII, no. 8, pp. 1173-1194, 1968.

[4] C. Micchelli, "The saturation class and iterates of the Bernstein polynomials," *Approx. Theory 8*, pp. 1-18, 1973.

[5] H. Oruc and N. Tuncer, "On the convergence and iterates of q-Bernstein polynomials," *Approx. Theory 117*, pp. 301-313, 2002.

[6] R. P. Kelisky and T. J. Rivlin, "Iterates of Bernstein polynomials," *Pacific J. Math. 21*, pp. 511-520, 1967.

[7] S. Ostrovska, "q-Bernstein polynomials and their iterates," *Approximation Theory 123*, pp. 232-255, May 2003.

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